SOLUTIONS

INTRODUCTORY CALCULUS, 2007

STRUCTURE OF PAPER

Question	Marks	Marks Obtained
1	9	
2	8	
3	13	
4	14	
5	12	
6	7	
7	15	
8	7	
9	7	
10	8	
11	9	
12	14	
13	5	
14	8	
15	8	
16	8	
17	12	
18	8	
19	8	

Total Marks = 180

1. [2, 3 & 4 = 9 marks]
Write an equation for each of the following linear functions.

(a)

x	-3	-1	1
y	-20	-10	0

$y = 5x - 5$

(b) The line perpendicular to $-2x - 3y = 12$ that cuts the y axis at $y = -3$.

$m_1 = -2/3$
$m_2 = 3/2$
$y = 3x/2 - 3$

(c) The tangent to the curve $y = \frac{-6}{x} + 2x$ at $x = -1$.

$y' = 6/x^2 + 2$

At $x = -1$ $y' = 8$

$m = 8$ and $(-1,4)$

$y = 8x + 12$

2. [2, 3 & 3= 8 marks]

If $g(x) = 2 - 3x - 2x^2$

Fully simplify your answer for:

(a) $g(0.5)$

$$\mathbf{2 - 3(0.5) - 2(0.5)^2 = 0}$$

(b) $g(a + 1)$

$$\mathbf{2 - 3(a + 1) - 2(a+1)^2}$$

$$\mathbf{= -2a^2 - 7a - 3}$$

(c) p, if $g(p) = -3$

$$\mathbf{-3 = 2 - 3p - 2p^2}$$

$$\mathbf{2p^2 + 3p - 5 = 0}$$

$$\mathbf{(2p + 5)(p - 1) = 0}$$

$$\mathbf{p = -5/2 \text{ or } p = 1}$$

3. [3, 4, 3 & 3 = 13 marks]

(a) Simplify the following expressing your answers with positive indices:

(i) $\dfrac{16\,a^2\,b^{-3}}{(2\,a\,b)^3} \times a\,b^5$

$$\frac{16\,a^2\,b^{-3}}{8a^3b^3} \times a\,b^5 = \frac{2}{b}$$

(ii) $\dfrac{6^{\,1-a}}{9^{\,a-1}} \times 18^{\,a-1}$

$$\frac{2^{\,1-a}\,3^{\,1-a}}{3^{\,2a-2}} \times 2^{\,a-1} \times 3^{\,2a-2}$$

$$= \mathbf{3^{\,1-a}}$$

(b) Solve the following, showing use of index rules:

(i) $(2x - 3)^{-2} = 0.25$

$$\mathbf{((2x - 3)^{-2})^{-0.5} = (0.25)^{-0.5}}$$
$$\mathbf{2x - 3 = 2}$$
$$\mathbf{x = 2.5}$$

(ii) $\text{Log}_x\,125 = 1.5$

$$\mathbf{x^{1.5} = 125}$$

$$\mathbf{(x^{1.5})^{2/3} = (125)^{2/3}}$$
$$\mathbf{x = 5^2 = 25}$$

4. [3, 5, 4 & 2 = 14 marks]

A storage container in the shape of a rectangular prism has a volume of $15m^{3.}$ It has a height of h, and the base length, l, is 50% longer than its width.

(a) Show how the formula $h = \frac{45}{2l^2}$ is determined from the volume.

$$\mathbf{V = l \times w \times h}$$
$$\mathbf{15 = l \times 2l/3 \times h}$$
$$\mathbf{h} = \frac{15 \times 3}{2 \times l \times l} = \frac{45}{2 \times l^2}$$

(b) The cost of materials for all sides is $30 per square metre.
Show that the cost of making the container could be determined by the formula:

$\text{Cost} = 40l^2 + \frac{2250}{l}$ **SA = 2LW + 2LH + 2WH**

$$\mathbf{= 2L \times 2L/3 + 2L \times 45/2L^2 + 2 \times 2L/3 \times 45/2L^2}$$
$$\mathbf{Cost = 30(\ 4L^2/3 + 45/L + 30/L)}$$
$$\mathbf{=40L^2 + 2250/L}$$

(c) What dimensions will minimize the cost of making the container? Give each dimension to the nearest cm.

$$\mathbf{C' = 80L - 2250/L^2}$$
$$\mathbf{0 = 80L - 2250/L^2}$$
$$\mathbf{L= 3.0441m}$$

L= 304cm, W=203cm, 243cm

(d) To the nearest dollar what is the minimum cost of making 100 such containers.

$$\mathbf{Cost = 100 \times (40 \times 3.0441^2 + 2250/3.0441)}$$

$$\mathbf{= \$\ 110979.65 = \$110980}$$

5. [3, 3, 3 & 3= 12 marks]

Solve the following equations exactly, where possible, otherwise to two decimal places. Working is expected.

(a) $2 \log 4 + \log x = \log 4 + \log (x - 2)$

log 16x = log 4(x-2)

16x = 4x -8

x = -2/3

but we can't have log(negative)

Therefore, no solutions.

(b) $\log (3x + 1) = 3$

$$\mathbf{3x + 1 = 10^3}$$

$$\mathbf{3x = 999}$$

$$\mathbf{x = 333}$$

(c) $\log_3 \frac{(3x+1)}{x} = -2$

$$\frac{(3x+1)}{x} = \mathbf{3^{-2}}$$

$$\frac{(3x+1)}{x} = \frac{1}{9}$$

$$\mathbf{27x + 9 = x}$$

$$\mathbf{x = -9/26}$$

(d) $\log_5 x^2 - \log_5 \frac{1}{x} = 6$

$$\mathbf{\log_5}\left(\frac{x^2}{\frac{1}{x}}\right)\mathbf{=6}$$

$$\mathbf{x^3 = 5^6}$$

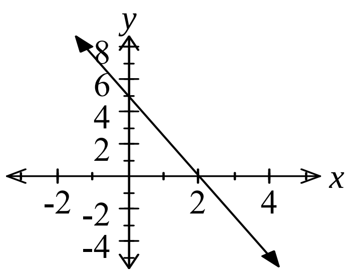
$$\mathbf{x = 25}$$

6. [2, 2 & 3 = 7 marks]

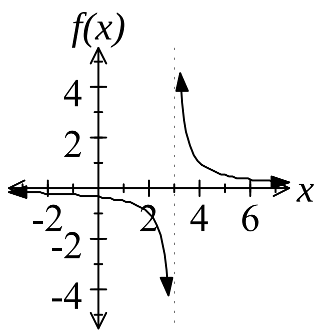
Determine the equation for the graphs below:

(a)

$y = 5 - 5x/2$

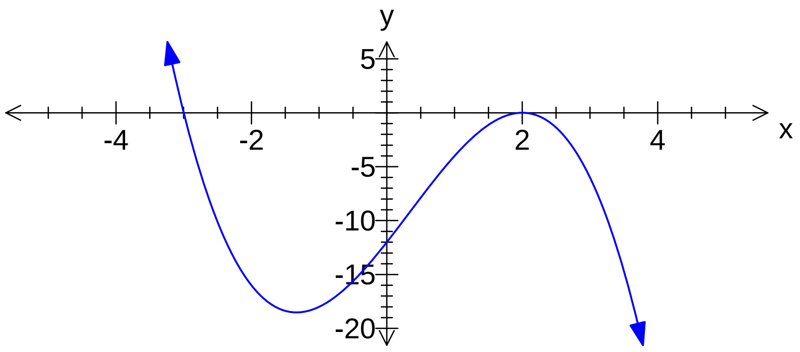


(b) **$y = 1/(x-3)$**



$y = -(x-2)^2(x+3)$

(c)



7. [3, 3, 4 & 5= 15 marks]

(a) Differentiate, with respect to x

$f(x) = (x^2 + 1)(x + 1)$

$\mathbf{2x(x+1) + 1(x^2 + 1)}$

$\mathbf{2x^2 + 2x + x^2 + 1}$

$\mathbf{3x^2 + 2x + 1}$

(b) Differentiate, with respect to x

$g(x) = \frac{1}{\sqrt{4x + 2}} = \mathbf{(4x + 2)^{-0.5}}$ $\mathbf{g' = -0.5 \times (4x + 2)^{-1.5} \times 4}$

$\mathbf{g' = -2/(4x+2)^{1.5}}$

(c) Differentiate, with respect to x

$h(x) = (3x + 2)^3\, 5\pi\, x$ $\mathbf{5\,\pi\,(3x + 2)^3 + 5\pi\ x\ 3\,(3x+2)^2\, 3}$

$= \mathbf{5\,\pi\,(3x + 2)^3 + 45\pi\ x\ (3x+2)^2}$

(d) (i) Differentiate, with respect to x

$k(x) = \frac{1 - 3x}{3x - 1}$ $k' = \frac{-3 \times (3x - 1) - 3\,(1 - 3x)}{(3x - 1)^2}$

$= \frac{-9x + 3 - 3 + 9x}{(3x - 1)^2}$

$= 0$

(ii) Explain the significance of your answer in (i).

k(x) can be written as k(x) $= \frac{-1(3x - 1)}{3x - 1} = -1$

and the derivative is hence 0.

8. [2, 3 & 2 = 7 marks]

The concentration, C mg/kL, of a chemical in the Roh river at time t days is given by:

$$C = e^{-2t}(5t + 1) \qquad 0 \le t \le 5$$

(a) State an expression for the instantaneous rate of change of C with respect to t.

$$\mathbf{C'(t) = -2e^{-2t}(5t+1) + e^{-2t}\,5}$$

$$\mathbf{= e^{2t}\,(3 - 10t)}$$

(b) Find the value of t when the instantaneous rate of change of C with respect to t is 0.

$$\mathbf{0 = e^{-2t}\,(3 - 10t)}$$

$$\mathbf{e^{-2t} = 0 \text{ or } (3 - 10t) = 0}$$

not possible t = 0.3days

(c) Find the maximum concentration of the chemical, to two decimal places.

t = 0.3 C = 1.37mg/kl

9. [7 marks]

Find the two points at which the tangent to the curve $y = \frac{x^3}{3} + \frac{5x^2}{4}$ is parallel to the line $-3x + 2y = 6$.

Gradient of line is 1.5.

$$y' = x^2 + 5x/2$$

$$1.5 = x2 + 5x/2$$

$$3 = 2x^2 + 5x$$

$$0 = 2x^2 + 5x - 3$$

$$0 = (2x - 1)(x + 3)$$

$$x = 0.5 \text{ or } x = -3$$

Therefore two points are (0.5, 17/48) and (-3, 2.25)

10. [3, 3 & 2 = 8 marks]
A marathon runner is attempting to run the Grand Canyon's desert event over a distance of 2600 km. On his first day he runs 55km, but fatigue and the desert heat causes him to run less each day. Each successive day he runs 1.5% less than the previous day.

(a) To the nearest km, how far does he run on the third and tenth day?

$r = 0.985$

Third day = 53 km

tenth day = 48 km

(b) After 30 days of running, how far short of completing the event is he?

$\text{Sum}_{30} = 1336.65\text{km}$

Still short = 1263 km

(c) On which day does he complete the event?

$$\mathbf{2600} = \frac{55 \times (1 - 0.985^t)}{(1 - 0.985)}$$

t = 81.697 days

Finishes on the 82nd day.

11. [4, 3 & 2 = 9 marks]
A particle moves in a straight line with acceleration given by the expression $a = 6t$ cm s^{-2} . The particles initial displacement is 1cm and its initial velocity is – 3 cm s^{-1}. The time t is in seconds and $t \geq 0$.

(a) Find expressions for the particles velocity and displacement.

a = 6t **$v = 3t^2 + c = 3t^2 - 3$**
$x = t^3 - 3t + k = t^3 - 3t + 1$

(b) Describe the motion of the particle when t = 2seconds.

t=2 **x= 3cm**
$v = 9cms^{-1}$
$a = 12ms^{-2}$
Particle is 3 cm to the right, moving to the right and accelerating.

(c) Find the time(s) when the particle is at the origin.

x = 0
$t^3 - 3t + 1 = 0$

t = 0.3473 sec or t = 1.5321 sec

12. [2, 3, 3, 3 & 3 = 14 marks]

Find:

(a) $\int \left(3x^2 - 2x + 0.2\right)dx$ **$x^3 - x^2 + 0.2x + c$**

(b) $\int (1-6x)^3 \, dx$ $\dfrac{-(1-6x)^4}{24} + c$

(c) $\int \left(\dfrac{-2}{(1-2x)^5}\right)dx$ $\dfrac{-1}{4(1-2x)^4} + c$

(d) $\int_0^2 e^{-3x-2} \, dx$ give an exact answer $\left(\dfrac{e^{-3x-2}}{-3}\right]_0^2$

$= \dfrac{e^{-8}}{-3} - \dfrac{e^{-2}}{-3}$

(e) $\int \dfrac{12x}{(3x^2-1)} \, dx$ **$2\ln(3x^2 - 1) + c$**

13. [5 marks]

A manufacturer's marginal cost function is:

$$\frac{dc}{dn} = 3n + 0.6n^2$$

If production is presently n = 200 units per month, how much more would it cost to increase production to 250 units per month?

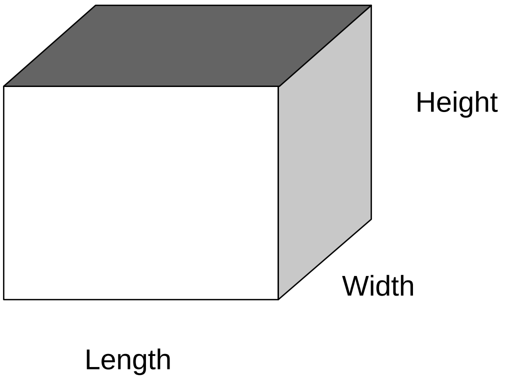
C = $\frac{3n^2}{2} + \frac{0.6\,n^3}{3} + c$

C(250) – C(200) = (3 ×250^2 ÷ 2 + 0.6 ×250^3÷3 + C) - (3 ×200^2 ÷ 2 + 0.6 ×200^3÷3 + C)

= (93750 + 3125000+C) –(60000+1600000+C)

= 1558750

14. [2, 3 & 3 = 8marks]
A glass fish container, without a lid, is in the shape of a rectangular prism with the width equal to the height. The total surface area of glass required is 5400 cm^2.



(a) Show that the length of the container can be calculated from the formula.

$$l = \frac{5400 - 2h^2}{3h}$$

$5400 = lw + 2lh + 2wh$
$5400 = lh + 2lh + 2hh$
$5400 = 3lh + 2h^2$

$$l = \frac{5400 - 2h^2}{3h}$$

(b) Now show that the volume of the container can is:

$$V = 1800\,h - \frac{2\,h^3}{3}$$

$$\mathbf{Vol = l\ w\ h}$$

$$= \mathbf{h\ h\ (}\frac{5400 - 2h^2}{3h}\mathbf{)}$$

$$= \frac{5400\,h^2}{3h} - \frac{2\,h^4}{3h}$$

$$= 1800\,h - \frac{2\,h^3}{3}$$

(c) Find the dimensions that will maximize the volume?

$$\mathbf{V' = 1800 - 2h^2}$$

$$\mathbf{0 = 1800 - 2h^2}$$

$\mathbf{h = 30}$ $\mathbf{w = 30}$ $\mathbf{l = 40}$

15. [2, 2 & 4 =8 marks]

A biologist is monitoring a fire ant infestation and notices that the area grows such that $\frac{dA}{dn} = 0.7A$ where **A** is the area in hectares and **n** is the number of weeks since the infestation was initially found. The initial infestation was estimated to be 1000 hectares.

(a) Find an expression for the area infested after **n** weeks. Use this formula to find the size of the infested area after 5 weeks.

$$\mathbf{A = 1000 \times e^{0.7t}}$$

$$\mathbf{t = 5 \quad A = 33115.45 \text{ hectares}}$$

(b) Use logarithms to find the time taken for the infestation to reach 5000 hectares.

$$\mathbf{1000 \times e^{0.7t} = 5000}$$

$$\mathbf{e^{0.7t} = 5}$$

$$\mathbf{0.7t \ln e = \ln 5}$$

$$\mathbf{t = \ln 5/0.7 = 2.3 \text{ weeks}}$$

(c) The local Environmental Council conducts a spraying operation to reduce the area on infestation. It predicts that the decay of the infestation after the spraying to be modeled by the equation $\frac{dA}{dn} = -0.5A$. If the area of infestation after 10 weeks was 9025hectares find to the nearest week when the spraying program commenced.

$$\mathbf{9025 = A_0 \times e^{-0.5\,(10)}}$$

$$\mathbf{A_0 = 1339428.76}$$

$$\mathbf{1339428.76 \times e^{-0.5t} = 1000 \times e^{0.7t}}$$

$$\mathbf{t = 6 \text{ weeks}}$$

16. [8 marks]
Consider the function $3y = 4x^3 - 3x + 6$.

Using calculus methods give the coordinates of all significant points (x and y intercepts, stationary points and points of inflection).

When x = 0 **y =2** **(0,2)**
When y=0 **$0 = 4x^3 - 3x + 6$** **(-1.36,0)**

$$\frac{dy}{dx} = 4x^2 - 1$$

$0 = 4x^2 - 1$

x= 0.5 or x = -0.5

x	0	0.5	1
$\frac{dy}{dx}$	-1	0	3

Min at (0.5, 5/3) **Max at (-0.5, 7/3)**

Second der. = 8x

0 =8x **Point of inflection at (0, 2)**

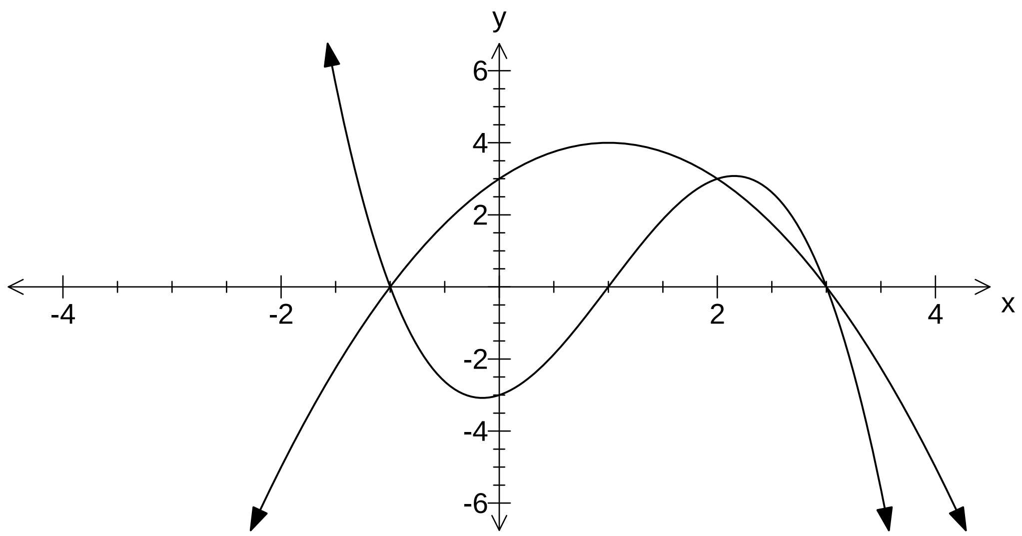
17. [2, 3, 2 & 5 = 12 marks]

Given $f(x) = -(x^2 - 1)(x - 3)$
And $g(x) = -x^2 + 2x + 3$ For $-3 \le x \le 4$

(a) Find all points of intersection.

(2,3) , (3,0) ,(-1,0)

(b) Provide a neat sketch of both graphs on the axes provided.



(c) For your graph give the values of x for which g(x) > f(x)?

$-1<x<2$, or $x>3$

(d) Use calculus methods to calculate the area between the two curves.

$g(x) - f(x) = x^3 - 4x^2 + x + 6$

$$\textbf{Area} = \left.\frac{x^4}{4} - \frac{4x^3}{3} + \frac{x^2}{2} + 6x\right]$$

$$= 11.25$$

$f(x) - g(x) = -(x^3 - 4x^2 + x + 6)$

Area = 0.583333

Total Area = 11.833333 units square.

18. [8 marks]

A curve has gradient function $f'(x) = kx + \frac{m}{x^2}$ where k and m are constants.
Find the function f(x) given that the graph passes through (2,-1) and has a turning point at (-2,0).

$$\mathbf{f(x) =} \frac{kx^2}{2} - \frac{m}{x} + c$$

And (2,-1) $\quad \mathbf{-1 = 2k -\frac{m}{2} + c}$ $\quad$ **Equation 1**

And (-2,0) $\quad \mathbf{0 = 2k + \frac{m}{2} + c}$ $\quad$ **Equation 2**

Take 2 from 1 $\quad$ **-1 = -m**
m = 1

Also when x =-2 the gradient function equals 0

0 = -2k + m/4

k = 1/8

19. [3 & 5 = 8 marks]

(a) Given $a + \frac{a}{3} + \frac{a}{9} + \ ... = 15$ Find a.

$T_1 = a$ **15 = a/(1 – 1/3)**

r = 1/3

Sum to Infinity = 15 **a =10**

(b) Given that:

1.5, 2x, 7x+ 3

are the first three terms of a geometric progression and that x is a positive value then determine the fifth term.

$(2x)^2 = 1.5 \times (7x + 3)$ **With x =3 1.5,6,24**

$4x^2 - 10.5x - 4.5 = 0$ **Term 5 = 384**

$8x^2 - 21x - 9 = 0$

(8x+3)(x-3)=0

x = -3/8 or x = 3