SOLUTIONS INTRODUCTORY CALCULUS REVISION ASSIGNMENT 4

PART A

1.

Graph	A	B	C	D	E	F	G	H
Function	VI	X	IV	V	II	VII	IX	VIII

2. (a) (i)

$$y = x^3 - 3x^2 + 3x + 4$$

$$\frac{dy}{dx} = 3x^2 - 6x + 3$$

$$\text{at } x = 2, \frac{dy}{dx} = 3$$

$\therefore$ slope of the tangent at x = 2 is 3

(ii) point (2, 6) m = 3

now $y - y_1 = m(x - x_1)$

ie $y - 6 = 3(x - 2)$

ie $y = 3x$

(b) $c(t) = 0.5t - 0.1t^2$

$$\frac{dc}{dt} = 0.5 - 0.2t$$

$$\text{let } \frac{dc}{dt} = 0 \Rightarrow 0.5 = 0.2t$$

ie $t = 2.5$

$$\text{now } \frac{d_2c}{dt^2} < 0 \ \therefore \text{ rel. maximum}$$

$\therefore$ at time = 2.5 minutes drug concentration is a maximum.

3. (i) $T_1 = 25\,000$

$T_2 = 18\,500$

$T_3 = 13\,690$

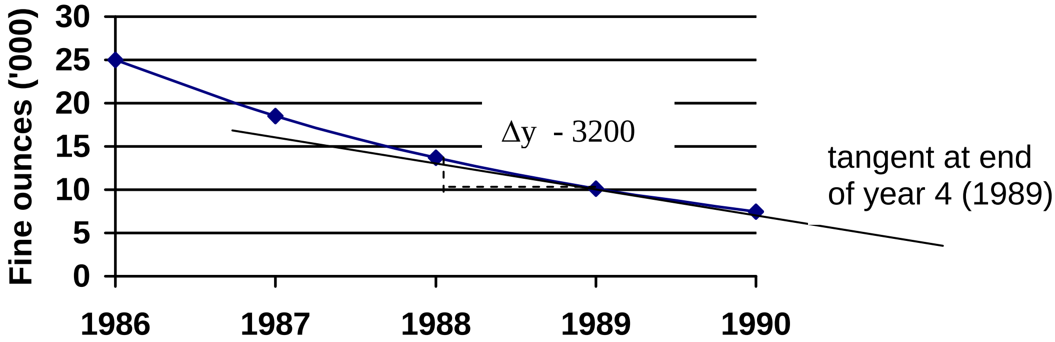
etc

$$\frac{T_2}{T_1} = 0.74 \text{ and } \frac{T_3}{T_2} = \frac{T_4}{T_3}$$

$\therefore$ geometric sequence, common ratio = 0.74

(ii)

KARRAMINE GOLD PRODUCTION



(iii) Inst. rate of change in fine ounces/year
$\approx$ - 3200 fine ounces/year.

(iv)

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$a = 25\,000$

$r = 0.74$
$n = 15$

$S_{15} = 95\,103$ ounces

(v)

$$S_\infty = \frac{a}{1 - r}$$

$$= \frac{25\,000}{1 - 0.74}$$

$$= 96\,154 \text{ ounces}$$

4. (a) (i) Function is positive and decreasing as x increases.
(ii) Function is positive; increasing, then stationary, then decreasing as x increases.
(iii) Function is negative and decreasing as x increases.

(b) x intercepts are 0, 6

(c) $y = -x^2(x - 6)$
ie $y = -x^3 + 6x^2$

(d) x coordinates are 0, 4

(e) at x = 0 there is a relative minimum on Graph I.
at x = 4 there is a relative maximum on Graph I.

(f) x coordinate is 2
at x = 2 there is a point of inflection on Graph I.

5.

$$P = \frac{10}{v} + v^2$$

$$\text{now } \frac{dP}{dv} = \frac{-10}{v^2} + 2v$$

$$\text{let } \frac{dP}{dv} = 0 \Rightarrow 2v = \frac{10}{v^2}$$

$$\text{ie } v^3 = 5$$

$$\therefore \quad v = \sqrt[3]{5}$$

$$\text{ie} \quad v = 1.71\,m/sec^2$$

$$\frac{d_2P}{dv^2} = \frac{+20}{v^3} + 2$$

$$\text{at } v = \sqrt[3]{5}\,,\ \frac{d_2P}{dv^2} > 0$$

$\therefore$ there is a relative minimum at v = 1.71 m/sec ^{2}

6. (i)

$$\frac{dy}{dx} = 2xe^{-x} + x^2e^{-x}(-1)$$

$$\text{ie } \frac{dy}{dx} = 2xe^{-x} - x^2e^{-x} = e^{-x}\left(2x - x^2\right)$$

(ii)

$$\frac{d_2y}{dx^2} = 2e^{-x} + 2xe^{-x}(-1) - \left[2xe^{-x} + x^2e^{-x}(-1)\right]$$

$$\frac{d_2y}{dx^2} = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$$

$$\frac{d_2y}{dx^2} = 2e^{-x} - 4xe^{-x} + x^2e^{-x} = e^{-x}\left(2 - 4x + x^2\right)$$

(iii)

$$\text{let } \frac{dy}{dx} = 0$$

$$\text{ie } 2xe^{-x} - x^2e^{-x} = 0$$

$$\text{ie } 2x - x^2 = 0 \quad \text{as } e^{-x} \neq 0$$

$$\text{ie } x = 0, 2$$

(iv)

$$\text{at } x = 0, \frac{d_2y}{dx^2} > 0$$

$\therefore$ local minimum at $x = 0$

and $y_{min} = 0$

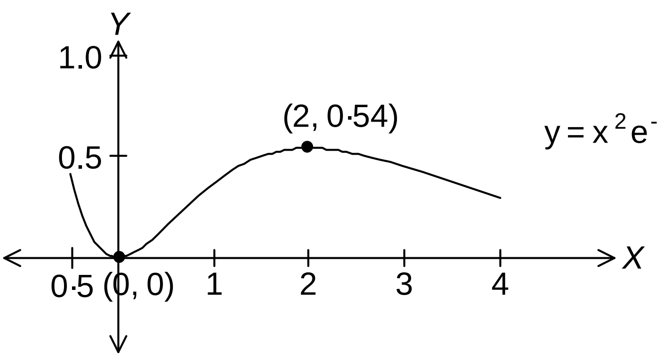
(v)

$$\text{at } x = 2, \frac{d_2y}{dx^2} < 0$$

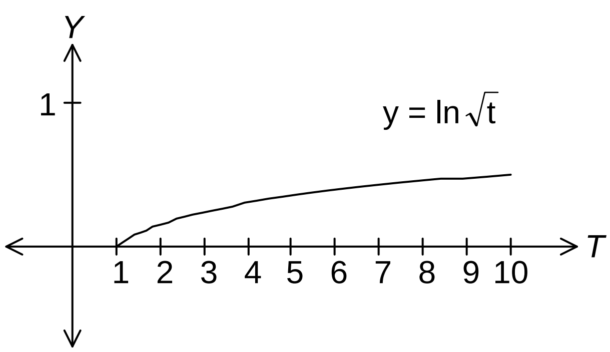
$\therefore$ local maximum at $x = 2$

and $y_{max} = 0.54$

(vi)



7. (i)



(ii)

$$y = \ln\sqrt{2.6}$$

ie $y = 0.478$ m

(iii)

$$y = \ln\sqrt{t}$$

ie $y = \ln t^{\frac{1}{2}}$

$$y = \frac{1}{2}\ln t$$

$$\therefore \frac{dy}{dt} = \frac{1}{2}.\frac{1}{t} = \frac{1}{2t}$$

(iv)

velocity = $\frac{dy}{dt} = \frac{1}{2t}$

$\therefore$ at t = 12 seconds, velocity = 0.042 m/sec

(v)

velocity = $\frac{1}{2t}$

$\therefore$ as t becomes very, velocity tends to zero.

8. (a)

If $m = Ae^{kt}$

then $\frac{dm}{dt} = Ae^{kt}.k$

ie $\frac{dm}{dt} = m.k$

ie $\frac{dm}{dt} = km$

ie $m = Ae^{kt}$ satisfies $\frac{dm}{dt} = km$

(i)

$m = Ae^{kt}$

when $t = 0, m = 110$

ie $110 = Ae^{k(0)}$

ie $A = 110$ gm

(ii) after 28 years

$55 = 110e^{k(28)}$

$0.5 = e^{28k}$

$28k = \ln 0.5$

ie $k = -0.0248$

(iii) at $t = 200$

$m = 110\, e^{0.0248\,(200)}$

$m = 0.78$ gm

9. (i)

$$\frac{dx}{dy} = e^y = x$$

(ii)

$$\frac{dy}{dx} = \frac{1}{e^y}$$

(iii)

$$\frac{dy}{dx} = \frac{1}{x}$$

(iv) $y = \ln x + c$

at $x = 1, y = 0 \Rightarrow c = 0$

ie $y = \ln x$

10. (i) $k = \ln 2 = 0.693$

$\therefore 2 = e^{0.693}$

(ii) $\therefore y = 2^t$

$= e^{0.693\,t}$

$$\therefore \frac{dy}{dt} = e^{0.693\,t}.0.693$$

or $= e^{\ln 2t}.\ln 2$

(iii) when $t = 5$, $\dfrac{dy}{dt} = 0.693 \text{ x } e^{0.693(5)}$

$= 22.16$

PART B

1.

Graph A	Function	(ix)
Graph B	Function	(ii)
Graph C	Function	(vii)
Graph D	Function	(i)
Graph E	Function	9vi)

2. $P(t) = 2t^3 - 15t^2 + 24t + 2$

(a)

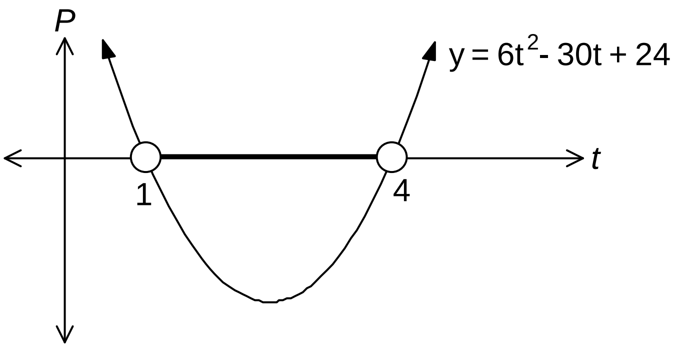
$$\frac{dP}{dt} = 6t^2 - 30t + 24$$

(b) Profit is decreasing if $\frac{dP}{dt} < 0$

ie $6t^2 - 30t + 24 < 0$

Method 1 Graphical

$$y = 6t^2 - 30t + 24$$
$$y = 6(t^2 - 5t + 4)$$
$$y = 6(t - 1)(t - 4)$$



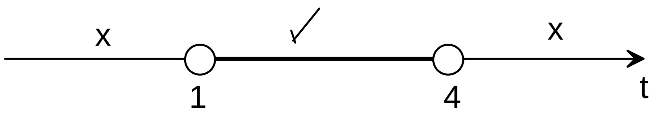
$\therefore$ Profit decreasing for $1 < t < 4$

OR

Method 2 **TEST INTERVALS**

$y = 6t^2 - 30t + 24$

$y = 6(t - 1)(t - 4)$



$\therefore$ Profit decreasing for $1 < t < 4$

3. (a) f(x) has a local maximum when x = 3

(b) f(x) has a local minimum when x = - 2

(c) f(x) has a point of inflection when x = 1 ($f^{/}(x) = 0$)

4. (a)

$$\text{If } f(x) = \frac{x^2}{x^2 - 1},\ f'(x) = \frac{2x(x^2 - 1) - x^2(2x)}{(x^2 - 1)^2}$$

$$= \frac{2x^3 - 2x - 2x^3}{(x^2 - 1)^2}$$

$$= \frac{-2x}{(x^2 - 1)^2}$$

$$\text{If } g(x) = \frac{1}{x^2 - 1},\ g'(x) = \frac{0.(x^2 - 1) - 1.(2x)}{(x^2 - 1)^2}$$

$$= \frac{-2x}{(x^2 - 1)^2}$$

$$\therefore f'(x) = g'(x)\ ;\ x \neq \pm 1$$

(b) As $f^{/}(x) = g^{/}(x)$, tangents to each of these curves at any value of x (except x $= \pm 1$) are parallel. Therefore the curves are "parallel" and must differ by a constant.

(Note: As a matter of interest f(x) = g(x) + 1)

5. (a)

If $P = Ae^{-kt}$

then $\dfrac{dP}{dt} = Ae^{-kt}.-k$

ie $\dfrac{dP}{dt} = -kP$

(b) (i) t = 0, P = 8090

Now $P = Ae^{-kt}$

ie $8090 = Ae^{-k(0)}$

ie $8090 = Ae^{0}$

ie $A = 8090$

(ii)

$P = 8090e^{-kt}$

when $t = 8$, $P = 5370$

$\Rightarrow 5370 = 8090e^{-8k}$

ie $\dfrac{5370}{8090} = e^{-8k}$

ie $0.6638 = e^{-8k}$

$\therefore -8k = \ln 0.6638$

ie $-8k = -0.4098$

ie $k = 0.0512$

(c)

$P = 8090e^{-0.05121}$

when $t = 12$, $P = 8090e^{-0.0512(12)}$

ie $P = 4376.41$

ie $P = 4380$ (to the nearest 10)

6. (a)

$\log_{10}(3^k),\ \log_{10}(9^k),\ \log_{10}(27^k), \ldots$

$= \log_{10} 3^k,\ \log_{10} 3^{2k},\ \log_{10} 3^{3k},\ \log_{10} 3^{4k}, \ldots .$

$= k\log_{10} 3,\ 2k\log_{10} 3,\ 3k\log_{10} 3,\ 4k\log_{10} 3, \ldots$

This is an arithmetic sequence with $T_1 = k\log_{10} 3$ and

common difference $d = k\log_{10} 3$

(b)

$3^k,\ 3^{k+1},\ 3^{k+2},\ 3^{k+3}, \ldots$

$= 3^k,\ 3^1.3^k,\ 3^2.3^k,\ 3^3.3^k, \ldots$

$= 3^k,\ 3.3^k,\ 9.3^k,\ 27.3^k,$

This is a geometric sequence with $T_1 = 3^k$ and common ratio = 3

$$\text{Hence } S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{3^k(1-3^6)}{-2}$$

$$= 364.3^k$$

7. (a) (i) If $u = \ln x$

then $\frac{du}{dx} = \frac{1}{x}$

(ii) If $y = u^3$

Then $\frac{dy}{du} = 3u^2$

(iii)

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

$$= 3u^2.\frac{1}{x}$$

$$= \frac{3(\ln x)^2}{x}$$

(b) (i)

If $y = xe^x$

then $\dfrac{dy}{dx} = e^x + xe^x$

(ii)

$$\frac{d^2y}{dx^2} = e^x + e^x + xe^x$$

$$= 2e^x + xe^x$$

(iii)

$$\frac{d^3y}{dx^3} = 3e^x + xe^x$$

hence , $\dfrac{d^ny}{dx^n} = ne^x + xe^x$

or $= (n + x)e^x$

8. (a) (i)

$$e^{3.5} = 1 + 3.5 + \frac{(3.5)^2}{2} + \frac{(3.5)^3}{6} + \frac{(3.5)^4}{24}$$

ie $e^{3.5} = 24.02$

(ii)

$$e^{-1} = 1 + (-1) + \frac{(-1)^2}{2} + \frac{(-1)^3}{6} + \frac{(-1)^4}{24}$$

ie $e^{-1} = 0.375$

(b) From calculator $e^{3.5} = 33.12$

$e^{-1} = 0.37$

In (a) (i) estimate is lower than calculator, but this is to be expected as additional terms would increase the approximation.

In (a) (ii) estimate is very close to calculator values. Again this is no surprise as additional terms are successively much smaller and alternate in sign.

(c)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...$$

$$\frac{d}{dx}\left(e^x\right) = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + ...$$

$$\text{ie} \quad \frac{d\left(e^x\right)}{dx} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$$

(d)

Note that $\frac{d}{dx}\left(e^x\right) = e^x$

9. (a) 7, 19, 52, … $T_1 = 7$, $r = \frac{19}{7} = \frac{52}{19} = \frac{141}{52} =$

ie $r = 2.72$

$\therefore T_{12} = ar^{11}$

$\approx 7(2.72)^{11}$

≈ 422042 bacteria

(b)

x:	1	2	3	4	5	6	7
y:	7	19	52	141	382	1039	2824
z:	1.95	2.95	3.95	4.95	5.95	6.95	7.95

(c) $z = x + 0.95$

There exists a linear relationship between z and x.

(d) From (c) $z = x + 0.95$

ie $\ln y = x + 0.95$

ie $y = e^{x + 0.95}$

OR From (a) $y = 7(2.72)^x$ OR $7e^x$

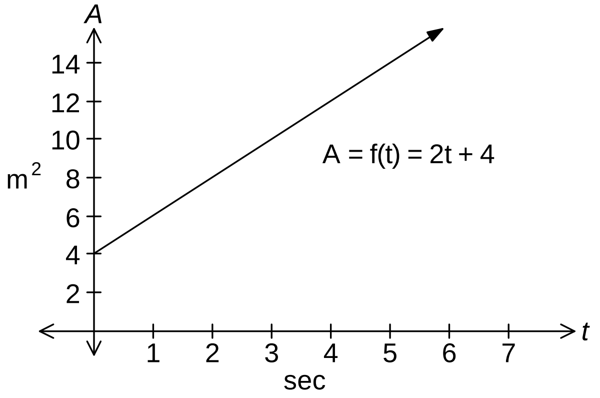
(e)

$$\frac{dy}{dx} = e^{x + 0.95}$$

At $x = 10$, $\frac{dy}{dx} \approx 56900$ bacteria / day

Can be estimated from graph.

10. (a)



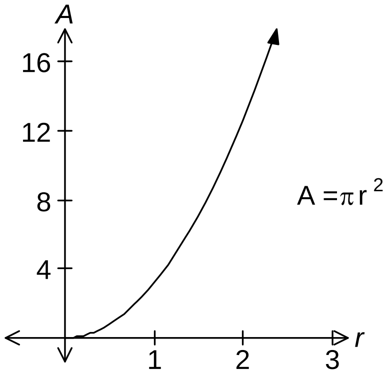
A = 2t + 4

This function has a constant rate of growth and is a LINEAR function.

(b)

$$\frac{dA}{dt} = \frac{d}{dt}(2t + 4) = 2$$

(c) $A = \pi r^2$

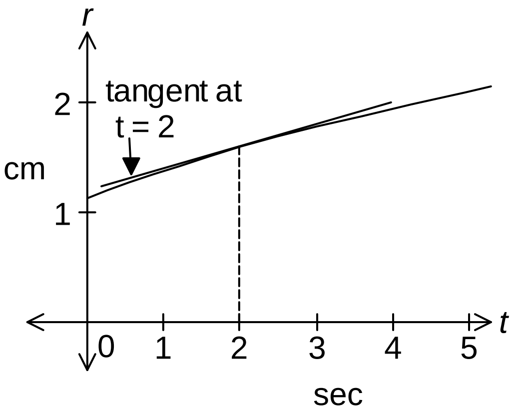


This is a quadratic function with increasing radius, A increases very rapidly.

(d) $\frac{dA}{dr} = 2\pi r$

(e)

t	1	2	3	4	5
A	6	8	10	12	14
r	1.38	1.6	1.78	1.95	2.11



(f)

$$\frac{dr}{dt} = \frac{dr}{dA}.\frac{dA}{dt}$$

$$= \frac{1}{2\pi r}.2$$

$$= \frac{1}{\pi r}$$

At $r = 2$; $\dfrac{dr}{dt} = \dfrac{1}{2\pi}$

ie $\dfrac{dr}{dt} = 0.16$ cm / sec

OR From graph $\dfrac{dr}{dt}$ = slope of tangent at r = 2

$$= \frac{0.7}{5} = 0.14 \text{ cm/sec } (\pm 0.3)$$