MODELLING FUNCTIONS

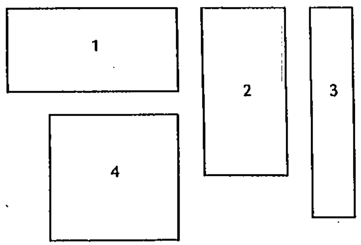
"INTRO CALC SURVIVAL KIT"

CAN YOU FIND THE RULE AS A FORMULA?

1 Shoe sizes were invented when lengths were measured in inches. Find the relationship between L and S.

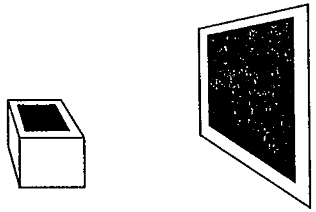
Length, L	9	10	11	12
Size, S	2	5	8	11

2 A number of rectangles are shown below. They each have a perimeter of 12 cm. What is the relationship between their height (h) and their width (w)?



3 Use the same rectangles for this problem. What is the relationship between the height (h) and area of the rectangles (A)?

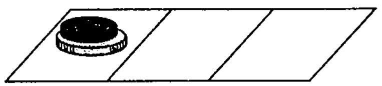
4 As you know, an overhead projector is designed to make a larger copy on a screen of the picture on its top.



The width of the picture (w cm) depends on how far away it is from the screen – the distance (d metres). Here are some figures for one projector.

Distance, d	1	2	3	4
Picture width, w	20	40	60	80

5 A good puzzle is called the Tower of Hanoi. There are three spaces on which coins of different sizes are placed.



The rules for the puzzle are these.

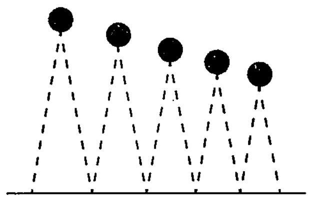
→ Start with two coins on one space as shown.
→ Move only one coin at a time.
→ Never put a larger coin on top of a smaller coin.

a You should be able to move the coins to another space in only three moves.
b Now try it with three coins.
c Now with four and even five coins.

What is the relationship between the number of coins and the number of moves needed?

6 If you drop a superball from a height of 200 cm it will rise up to 90% of its height on the next bounce. Then it rises to 90% of its height on the next bounce, and so on. Really neat!

What is the relationship between the number of the bounce (n) and height of the bounce (h)?



7 You can tell the length of a pendulum (L cm) if you know the time it takes for a to-and-fro swing (t seconds). What is the relationship between t and L?

Time, t (seconds)	1	2	3	4
Length, L (cm)	25	100	225	400

8 You try to throw a balloon, but it doesn't go far due to air resistance. The air resistance depends on the surface area.

The surface area (A) depends on the radius of the balloon (r). How?

Radius, r (cm)	1	5	8	12
Area, A (cm^2)	13	314	804	1809

9 For many years inflation caused the price (p cents) of a stamp for a simple letter to rise. The table gives the price for some numbers of years after 1966.

Years after 1966, t	0	8	15	20
Price, p (cents)	6	10	24	33

10 A cup of coffee is made from boiling water. The person who made it forgets about it and it cools down. The table shows that there is a relationship between the time (n minutes) and the temperature (T°C).

Time, n (minutes)	0	2	5	10
Temperature, T°	100	85	67	48

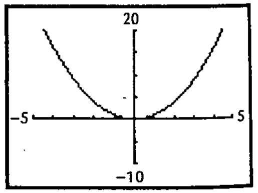
11 You can estimate your reaction time if you let someone drop a ruler between your fingers for you to catch. There is a relationship between the distance the rule falls (d centimetres) and the time (t seconds). Here is the data. What is the relationship? Try it!

Time, t (seconds)	0.09	0.16	0.2	0.24
Distance, d (cm)	4.0	12.5	19.6	28.2

FOR THE ANSWERS YOU HAVE TO ASK MR C. :)

QUADRATICS 1

1 The graph shows the function $y = x^2$. On your own grids sketch the graph of both $y = x^2$ and the function given. You should be able to do this without a graphics calculator. ABC



a $y = x^2 - 1$　　b $y = (x-3)^2$
c $y = (x+2)^2 + 3$　　d $y = -x^2$
e $y = 3x^2$　　f $y = (x-1)^2 + 1$

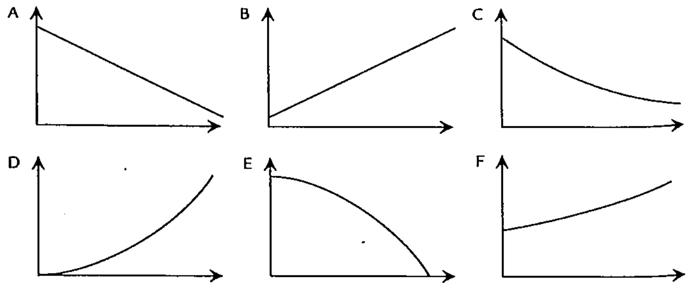
2 Sketch the graph of each function, marking its intercepts and naming its vertex. ABC

a $y = x^2 - 4$　　b $y = (x-2)^2$　　c $y = (x+2)(x-3)$
d $y = 4 - x^2$　　e $y = x^2 + 2$　　f $y = (x-1)(x-1)$

3 A car is accelerating from the traffic lights. The speed of the car (km/h) and the distance it has gone (in metres) are related by the formula: $d = 0.001s^2$.

a Complete a table of values for s from 0 to 60 in steps of 10.
b Sketch a graph of the relationship. ABC

4 a Which of the six graphs below show growth? Which show decay?
b Which are linear? Which are exponential? Which are quadratic? ABC



5 Explain the differences between the six types of function in question **4.** ABC

6 Each of the tables below is either linear, exponential or quadratic. Select which is which by examining the patterns in the data. ABC

a

x	0	1	2	3
y	1	3	5	7

b

x	0	1	2	3
y	9	8	5	0

c

x	0	1	2	3
y	9	6	3	0

d

x	0	1	2	3
y	4	2	1	0.5

e

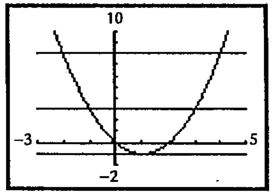
x	0	1	2	3
y	1	4	9	16

f

x	0	1	2	3
y	2	4	8	16

7 Use a graphics calculator and regression to find the formula for each of the tables in question **6.** ABC

8 Use the graph to write down the exact solutions of each equation. ABC



a $x^2 - 2x = 0$　　b $x^2 - 2x = 8$
c $x^2 - 2x - 3 = 0$　　d $x^2 - 2x + 1 = 0$

9 Expand each of the following. BC

a $(x+5)(x+2)$　　b $(x-2)(x+1)$
c $(2x-1)(3x-4)$　　d $(3x-2)^2$
e $(x-3)(x+3)$　　f $(4x-3)(x-2)$

10 Factorise each of the following. BC

a $x^2 + 4x + 3$　　b $x^2 - 4x + 3$　　c $x^2 - 4x - 12$
d $x^2 - 4x + 4$　　e $x^2 - 400$　　f $x^2 + 4x - 21$

11 Use the null factor law to solve these quadratic equations. BC

a $(2x-5)(x+3) = 0$　　b $x^2 - x = 0$　　c $x^2 - x = 2$
d $x^2 - x - 6 = 6$　　e $x^2 - 3x - 10 = 0$　　f $x^2 - 9x + 18 = 0$
g $x^2 + 18x + 81 = 0$　　h $x^2 - 13x + 30 = 0$　　i $x^2 + 27x + 50 = 0$

12 Find the exact solutions of these quadratic equations. Use surds if needed.

a $x^2 = 5$　　b $(x-2)^2 = 0$　　c $(x+2)^2 = 9$　　d $2x^2 = 7$ BC

13 Say which of these quadratic equations has no solutions. Then solve the rest, giving answers to two decimal places. Use a graphics calculator to help.

a $x^2 + 2x = 2$　　b $x^2 + 2x + 2 = 0$　　c $x^2 = 2x + 2$ BC

14 A piece of land is in the shape of a right-angled triangle. The hypotenuse is 4.5 m longer than the shortest side, and the remaining side is 1 m shorter than the hypotenuse. Find the lengths of the sides of the triangle. BC

15 Use quadratic regression, or other means, to find each formula. BC

a

x	0	1	2	3
y	1	4	5	4

b

x	0	1	2	3
y	−9	−8	−5	0

c

x	0	1	2	3
y	2	1	2	5

16 Simplify the surd expressions (no surds in the denominators of answers). BC

a $3\sqrt{3} - \sqrt{27}$　　b $\dfrac{\sqrt{72}}{\sqrt{27}}$　　c $\dfrac{5\sqrt{2}}{2\sqrt{5}}$
d $(\sqrt{5} - \sqrt{2})^2$　　e $(\sqrt{27} - \sqrt{3})(\sqrt{27} + \sqrt{3})$　　f $\sqrt{18}(\sqrt{8} + \sqrt{2})$

QUADRATICS 2

1 Express each quadratic expression as a perfect square and an additional term. Check your work with a graphics calculator. Find the value of x that gives the minimum of the function and the value of the minimum.

a $x^2 + 2x + 5$ b $x^2 - 2x - 5$ c $x^2 - 2x - 1$
d $x^2 + 2x + 1$ e $x^2 + 4x + 5$ f $x^2 - 4x - 5$
g $x^2 - 4x - 1$ h $x^2 + 4x + 1$ i $x^2 + 6x + 5$
j $x^2 - 6x - 5$ k $x^2 - 6x - 1$ l $x^2 + 6x + 1$
m $x^2 + 8x + 5$ n $x^2 - 8x - 5$ o $x^2 - 8x - 1$

2 How does the position of the vertex depend on:
a the constant (last) term in the trinomial?
b the sign (+ or −) of the middle term?
c the value of the middle term?

3 Find the solutions to each quadratic equation by completing the square.

a $x^2 + 2x + 5 = 0$ b $x^2 - 2x = 5$ c $x^2 - 2x = 1$
d $x^2 + 2x + 1 = 0$ e $x^2 + 4x + 5 = 0$ f $x^2 - 4x = 5$
g $x^2 - 4x = 1$ h $x^2 + 4x + 1 = 0$ i $x^2 + 6x + 5 = 0$
j $x^2 - 6x = 5$ k $x^2 - 6x = 1$ l $x^2 + 6x + 1 = 0$
m $x^2 + 8x + 5 = 0$ n $x^2 - 8x = 5$ o $x^2 - 8x = 1$

4 Find accurate or approximate (to one decimal place) solutions to each equation using the quadratic formula. Check with a calculator program.

a $x^2 + x - 2 = 0$ b $x^2 - x - 2 = 0$ c $x^2 + 3x + 2 = 0$
d $x^2 - 3x - 2 = 0$ e $x^2 + 5x + 2 = 0$ f $x^2 - 5x - 2 = 0$
g $x^2 - 7x - 1 = 0$ h $x^2 + 7x + 1 = 0$ i $2x^2 + 3x - 5 = 0$
j $2x^2 - 5x - 5 = 0$ k $2x^2 + 7x + 5 = 0$ l $2x^2 - 9x - 5 = 0$
m $3x^2 + 4x - 5 = 0$ n $3x^2 - 4x - 5 = 0$ o $3x^2 - 5x - 1 = 0$

5 Express each quadratic expression as a perfect square and an additional term. Check your work with a graphics calculator. Find the value of x that gives the minimum of the function and the value of the minimum.

a $x^2 + x + 2$ b $x^2 - x - 2$ c $x^2 - x - 1$
d $x^2 + x + 1$ e $x^2 + 3x + 2$ f $x^2 - 3x - 2$
g $x^2 - 3x - 1$ h $x^2 + 3x + 1$ i $x^2 + 5x + 2$
j $x^2 - 5x - 2$ k $x^2 - 5x - 1$ l $x^2 + 5x + 1$
m $x^2 + 7x + 2$ n $x^2 - 7x - 2$ o $x^2 - 7x - 1$

6 Find exact solutions to each quadratic equation by completing the square or using the quadratic formula.

a $x^2 + x + 1 = 0$ b $x^2 - x - 2 = 0$ c $x^2 + 3x + 2 = 0$
d $x^2 - 2 = 0$ e $x^2 + 5x + 2 = 0$ f $x^2 - 5x - 2 = 0$
g $x^2 - 7x - 1 = 0$ h $x^2 + 7x = 0$ i $2x^2 + 3x + 5 = 0$
j $2x^2 - 5x - 5 = 0$ k $2x^2 + 7x + 5 = 0$ l $2x^2 - 5 = 0$
m $3x^2 + 2x + 5 = 0$ n $3x^2 - 4x - 5 = 0$ o $3x^2 - 5x - 1 = 0$

7 You throw a ball upwards from ground level at v m/s. Immediately it starts to slow, then eventually stops, and falls. The height of the ball (h metres) after t seconds is given by the formula $h = t(v - 5t)$. Use the values of v given in the table.

Use v =	50	40	30	20	10
Formula h =	$t(50 - 5t)$	$t(40 - 5t)$	$t(30 - 5t)$	$t(20 - 5t)$	$t(10 - 5t)$

a Complete the table until the ball lands on the ground. Use a graphics calculator to help. Use a different table and formula for each value of v in the table above.

t	0	1	2	3 ...
h	0			

b Sketch the graph for each speed, v. Use a WINDOW of $0 \le x \le 10$ and $0 \le y \le 130$ and enter the functions so that the family of graphs is shown.
c Complete the square and use it, or the graph, to find the time when the ball reaches its maximum height and the height it reaches. Repeat for each speed.
d Find and explain the pattern in the times which were the answers you found in part c.
e Find the two times when the ball is at exactly half of its maximum height.
f Try to find the pattern in the times for the answers to part e. Use surds to help prove it.

8 A ball thrown at 45° goes further than a ball thrown at any other angle, if the speeds are the same and air resistance is ignored.
The path of the ball is a parabola, and has this formula:

$$y = x - \frac{10x^2}{v^2} \quad \text{for any initial speed, } v$$

Use v =	50	40	30	20	10
Formula y =	$x - \frac{x^2}{250}$	$x - \frac{x^2}{160}$	$x - \frac{x^2}{90}$	$x - \frac{x^2}{40}$	$x - \frac{x^2}{10}$

a Complete the table until the ball lands on the ground. Use a different table and formula for each value of v above. A graphics calculator will help.

x	0	5	10	15...
y	0			

b Sketch the graph for each speed. Use a WINDOW of $0 \le x \le 250$ and $0 \le y \le 165$ and enter the functions so the family of graphs is shown.
c Complete the square and use it, or the graph, to find the point (x, y) at which the ball reaches its maximum height. Repeat for each speed.
d Try to find the pattern in the answers to part c.
e Find how far the ball goes horizontally. Repeat for each speed.
f Try to find the pattern in the answers to part e.

9 For a ball thrown into the air vertically with some initial speed, its height at any moment can be related to its speed. The relationship is quadratic and has this formula: $y = 0.05(v^2 - x^2)$. The initial speed is v m/s, the speed at any moment is x m/s and the height is y metres.

v	50	40	30	20	10
y	$0.05(2500 - x^2)$	$0.05(1600 - x^2)$	$0.05(900 - x^2)$	$0.05(400 - x^2)$	$0.05(100 - x^2)$
x	−50 to 50	−40 to 40	−30 to 30	−20 to 20	−10 to 10

a Complete five tables until the ball lands on the ground. Use the range of x values suggested. Use a graphics calculator to help.
b Sketch the graph for each speed. Use a WINDOW of $-50 \le x \le 50$ and $0 \le y \le 130$ and enter the functions so the family of graphs is displayed.
c Why does the graph have a maximum when $x = 0$? What does this point mean in reality?
d Why is the graph symmetrical about the y-axis? What does a negative x value mean in reality?
e For each graph, find the maximum value.
f Try to find the pattern in the answers to part e. Prove it.
g Why is a restricted range of x-values needed for each graph? What does going outside this range mean in reality?

EXPONENTIAL GROWTH AND DECAY

1. During a bike ride from Sydney to Perth, the temperature changed a lot. The motorist accompanying the cyclist kept a record of the average hourly temperature and how much the cyclist drank in each hour.

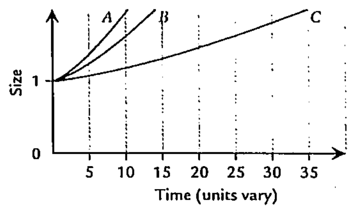
Temperature (°C)	10	15	20	25	30	35
Volume (mL)	50	100	200	400	800	1600

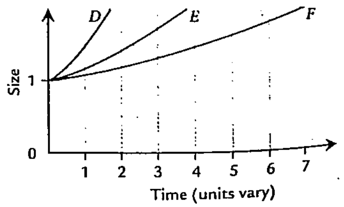
a Predict the water needed per hour when the temperature is 40°C.
b What temperature rise seems to cause a doubling of water needs?
c Find a formula for the relationship.
d The cyclist drank 250 mL in 1 hour. What was the average temperature?

2. Since the 1970s the number of serious assaults per 100 000 people in Australia has doubled each 7 years. In 1973 it was 21.
a In which year was it 42? In which year was it 84?
b Find a formula for the relationship, where n represents the number of years since 1973.
c Does your formula give the same results as $A = 21(2)^{\frac{n}{7}}$? Explain.
d Use a formula to find the year when this model predicts that A reached (or will reach) 300.

The growth rate of many living things is exponential for some time near the start of their lives. The table gives some growth rates. The graphs represent the growth up to doubling the initial size.

Life form	Bacteria	Algae	Wheat	Kitten	Duckweed	Ape
Growth rate	2%/minute	7%/h	5%/day	10%/week	50%/week	20%/month





3. a Match each life form to one of the graph curves.
b Write the formula for each growth.
c Read the doubling time from the graph and check with your calculator.
d Use the results to complete this table.

Growth rate (%)	2	5	7	10	20	50
Doubling time						

e Find the relationship between growth rate and doubling time. Suggestion: enter the values into the statistical lists and try different regression types until one fits rather well.

4. Some early postage rates for letters since the introduction of decimal currency are given in the table.

Year	1966	1971	1976	1981	1986
Years since 1966	0	5	10	15	20
Cost in cents	4	7	18	24	36

a Work out the percentage increases for each 5-year period. Take an average. Write it as the base number for an exponential formula.
b Find an annual rate that, over 5 years, produces this average 5-year increase.
c You should have something like 1.122 as the base number. What annual inflation rate does this price increase represent?
d Use the formula to compare the model's predictions with the real values.
e Predict the cost of a stamp in 1996 using the model (actually 45 cents).
There is a danger in extrapolation. Be careful when using a model past the data set for which it was created.

The energy released in an earthquake is measured on the Richter scale. The Richter number (r) is a factor in the power term in the formula for the energy released. Here is some data.

Richter number (r)	0	2	4	6
Energy (units)	4.8	4800	4.8×10^6	4.8×10^9

5. a Use the table to find a formula.
b Compare your formula with $4.8 \times 10^{1.5r}$. Explain why they are the same.
c What is the effect of a rise of 2 on the Richter scale?
d Find the energy of the 1989 Newcastle earthquake, which had $r = 5.6$.
e Find the energy of the largest earthquake in Australia, in Tennant Creek, 1988, measuring 8.0.
f How many times more energetic was the Tennant Creek earthquake than the Newcastle one?

6. The half-life of a decaying element is given. Find the exponential function giving the amount left for a 100-mg sample after x units of time.
a Plutonium: 240 000 years b Polonium 218: 3.05 minutes
c Thallium: 1.3 minutes d Protactinium 234: 1.14 minutes

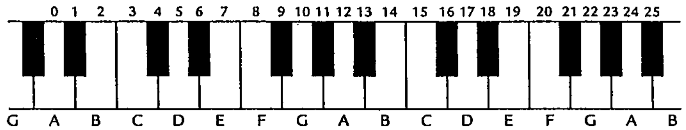
7. If you use your calculator to find certain values with negative base numbers you may get an error! Try these. ('Correct' values are in the answers.)
a (−1)^0.5 b (−1)^0.6 c (−1)^0.7 d (−1)^0.8

8. Graph these on your graphics calculator. Only the 'real' values are calculated as can be seen in the TABLE and the graphs look interesting on the screen.
a Y=(−1)^X b Y=(−0.9)^X c Y=(−0.5)^X d Y=(−1.5)^X

9. Find a suitable formula to represent each situation. Invent your own variables and values.
a A paintbrush is washed and each time loses 10% of the paint left after the last wash.
b With inflation the value of the dollar falls, buying 10% less each year.
c The chemical pollution downstream from a factory outlet drops 4% for each kilometre downstream.
d The temperature of the atmosphere falls by 20% for each 5 km of height above sea level.

GUITAR FRETS

On our modern keyboard, there are 12 notes from one A to the next. The step from one note to the next is called a semitone.



The pitch of each note is given in a number, called its frequency, to tell us how high or low it is.

We use the formula $f = 110 \times 2^{\frac{x}{12}}$ to find the frequencies for each semitone.

1 Use this formula to find the frequencies (to three decimal places).

Note	0(A)	2(B)	4(C#)	5(D)	7(E)	9(F#)	12(A)
Frequency	110						

2 Pythagoras (yes, him!) thought that there was a whole number ratio relating the frequencies of the notes A, C# and E. Divide the frequencies of those notes by 27.5 to find the ratio he discovered. You will see that he was very close indeed!

3 Explore other ratios of frequencies in the table. Some are very close to whole numbers.

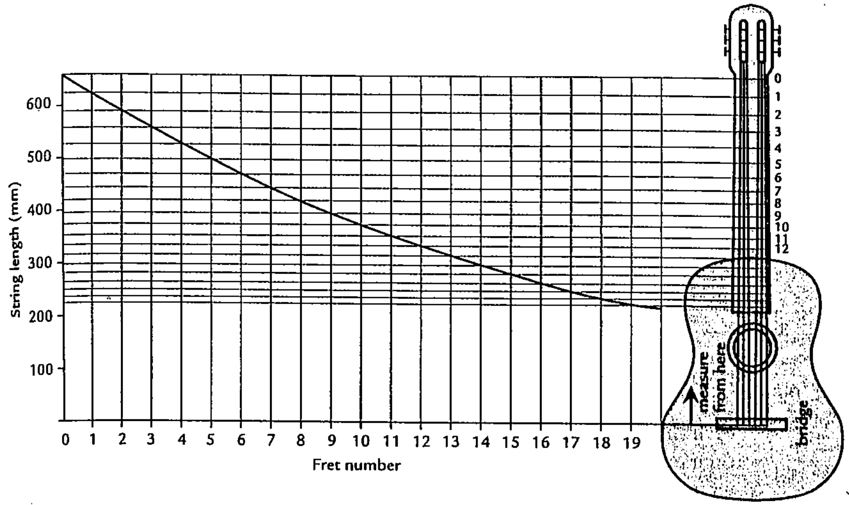
On a guitar, the frets are the strips that cross the finger board at right angles to the strings. A player pushes the string onto the fret just behind the fret position so that the length of string that vibrates is from the bridge to the fret used (see the diagram of guitar opposite).

The frets get closer together as the string gets shorter. There is a mathematical formula that relates the position of the fret to the number of the fret. This investigation will help you to find it.

4 Call the open string fret 0. If you have a guitar, measure the fret distances carefully. Here are some distances on my guitar.

Fret number x	0	1	2	3	4	5	6	7	8	9
Distance (mm)	660	624	590	558	527	497	470	444	420	396
Ratio	–	0.944								

Fret number x	10	11	12	13	14	15	16	17	18	19
Distance (mm)	375	353	334	316	298	282	266	251	238	225
Ratio										



5 Find the ratio of each pair of neighbouring fret distances. They won't all be exactly the same, but close; you could take an average.

6 Is the string length either exactly, or almost, half as long for fret 12 as it is for fret 0? Check this for other fret distances that differ by 12, such as 13 and 1, 14 and 2, and so on.

7 Since the ratio is the same for each fret, and 12 such ratios make 0.5, compare the ratio you got with the 12th root of 0.5, that is $0.5^{\frac{1}{12}}$.

The formula for the string length is:

$$L = (\text{open length}) \times 0.5^{\frac{x}{12}}$$

where x is the fret number.

8 Use this formula to check the values you measured.

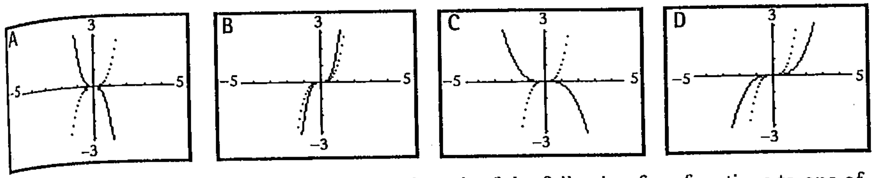
The model will be a little inaccurate. This is because the string is stretched slightly when it is pushed onto the fret, so the lengths are correspondingly a little longer than the theoretical model predicts.

CUBIC FUNCTIONS

See example 22

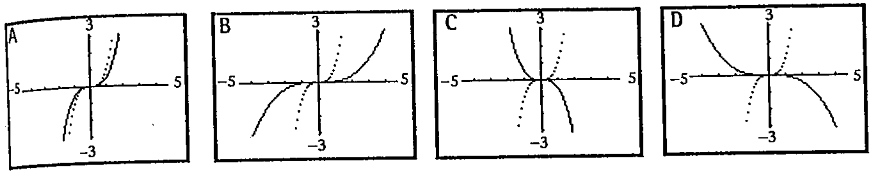
1 The five curves are cubic functions with different coefficients of x^3. The function $y = x^3$ is dotted on each of the four grids below. Match each of the following four functions to one of the other curves shown as solid lines on each grid.

a $y = 0.2x^3$ b $y = -0.2x^3$ c $y = 2x^3$ d $y = -2x^3$



2 Repeat question 1 to match each of the following four functions to one of the graphs below.

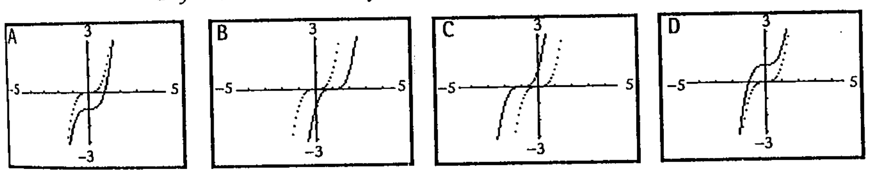
a $y = -0.5x^3$ b $y = 0.05x^3$ c $y = 0.5x^3$ d $y = -0.05x^3$



See example 23

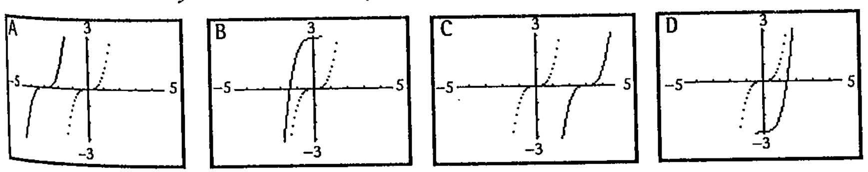
3 Four cubic functions are moved, either up, down, left or right, away from $y = x^3$, which is shown dotted on each of the four grids below. Match each of the following four functions to one of the other curves shown as solid lines on each grid.

a $y = x^3 + 1$ b $y = x^3 - 1$ c $y = (x - 1)^3$ d $y = (x + 1)^3$



4 Repeat question 3 to match each of the following four functions to one of the graphs below.

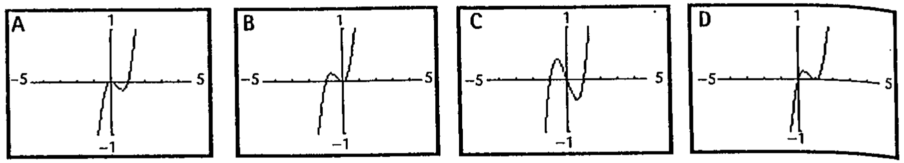
a $y = x^3 + 3$ b $y = x^3 - 3$ c $y = (x - 3)^3$ d $y = (x + 3)^3$



See example 24

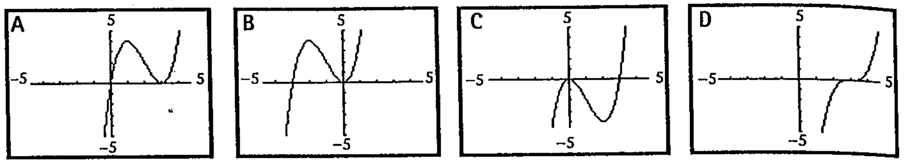
5 The four graphs shown cross or touch the axis at clearly marked positions. Match the four functions to the graphs.

a $y = x(x - 1)^2$ b $y = x^2(x - 1)$ c $y = x(x - 1)(x + 1)$ d $y = x(x + 1)^2$

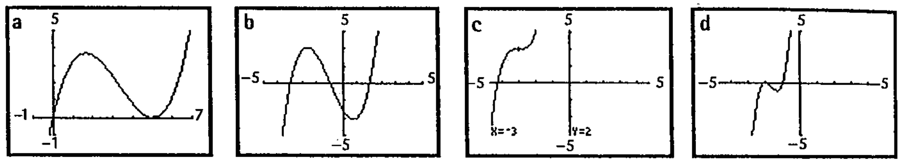


6 Repeat question 5 to match these functions and graphs.

a $y = x(x - 3)^2$ b $y = x^2(x - 3)$ c $y = (x - 3)^3$ d $y = x^2(x + 3)$



7 Name the functions whose graphs are shown. The coefficient of x^3 is 1.

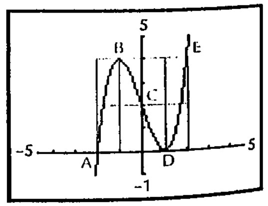


8 Without using your graphics calculator, sketch the graph of the function, showing the y- and x-intercepts. Then check with the graphics calculator, and fix any errors.

a $y = x(x + 1)^2$ b $y = (x + 1)^2(x - 2)$ c $y = (x + 3)(x + 2)(x + 1)$
d $y = x(x + 5)^2$ e $y = x(x - 4)^2$ f $y = (x - 1)^2(x - 2)$
g $y = x^3 - 8$ h $y = 8 - x^3$

9 **Did you know?**

For every cubic function of the 'hill and valley' type, there is a simple rectangle to help you sketch the curve. There are five special points which are evenly spaced horizontally. For each cubic curve of this shape these five points will always be placed as shown by the coloured grid in the diagram.



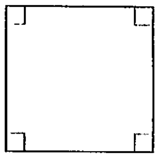
Investigate this further by graphing each of these functions.

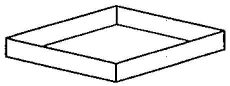
a $y = x^3 - 3x$, lines at $y = \pm 2$ b $y = x^3 - 12x$, lines at $y = \pm 16$
c $y = x^3 - 3x^2 + 2$, lines at $y = \pm 2$ d $y = x^3 - 6x^2 + 9x - 2$, lines at $y = \pm 2$

MAXIMUM VOLUME

A paper box

Imagine a square of paper, 24 cm on one side. You are going to cut a square from each corner, and fold up the remaining sides so that it forms a square box without a lid. Imagine cutting a 2-cm square from the corners.



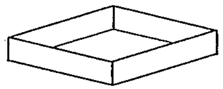


1 How high is the box?

2 When you fold up the sides of the box the base will no longer be 24 cm square. How wide is the base?

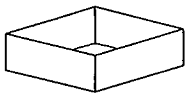
3 Work out the volume of the box.

4 Now let's cut a 3-cm square from each corner. Fold up it to make the box again.



What is the height? What is the base? What is the volume? Is the volume larger or smaller?

5 Now try a 5-cm square cut from the corner.



What is the height? What is the base?
What is the volume?
Is the volume larger or smaller?
The problem is: what is the size of square that will give the greatest (maximum) volume?

6 Complete this table of your results.

Square	1	2	3	4	5	6	7	8	9	10
Height		2								
Length		20								
Width		20								
Volume		800								

7 What seems to be the greatest volume?

8 What size of square gives the greatest volume?

9 Can you be sure that no greater volume is possible, with another size of square?

A formula

You could keep using trial and error, but here is another method, making use of your graphics calculator.

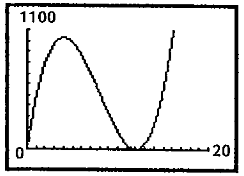
We need a formula that will tell us the volume for any size of square cut from the corners. To achieve it you must generalise the method you have been using to find the volume. Let x cm be the height of the box.

10 What expression gives the length of the square base?

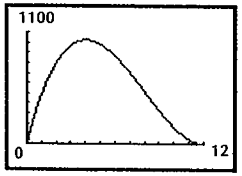
11 Here is an expression for the volume.
Volume = height × base area

$$y = x(24 - 2x)^2$$

Graph this using $0 \le x \le 20$ and $0 \le y \le 1100$.



12 What is going on for larger squares? Is it possible to cut out larger squares than 12 cm? Since these values make no sense at all, we say they are outside the domain of the function. This means we only consider x-values between 0 and 12. The others make no sense.
Here is the graph drawn only between 0 and 12.



13 As you can see, the graph reaches a maximum and then drops back to zero. The zero value is 12. What is the base length of the box when the cut-out square has a side of 12 cm? Explain why the volume is zero.

14 What value of x gives the maximum? Use either TRACE or CALC maximum.

15 Now solve these similar problems. Each paper shape described is used as before to fold into a box without a lid. What size corner square gives the maximum?

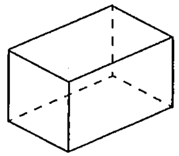
- **a** a square of side 6 cm
- **b** a square of side 12 cm
- **c** a rectangle of side 16.5 cm by 20 cm
- **d** a rectangle of size 25 cm by 40 cm

A parcel

To be accepted by a courier service, a parcel of square cross-section must meet the following condition.

The total of the smallest circumference of the parcel and its longest dimension must not exceed the length given. You can have a long thin parcel or a shorter fat parcel.

16 For each length, what are the dimensions of the parcel that fits the conditions and also gives the greatest volume?

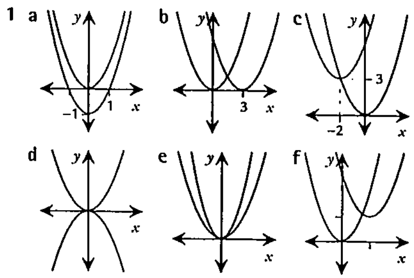


- **a** total = 180 cm
- **b** total = 240 cm
- **c** total = 216 cm
- **d** total = 420 cm

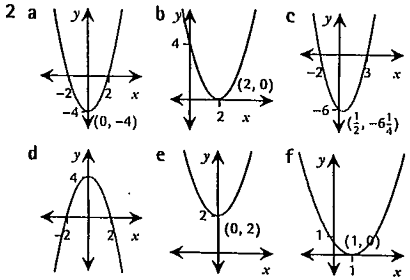
SOLUTIONS 1 — INTRO CALC SURVIVAL KIT

Quadratics 1

1 a b c d e f



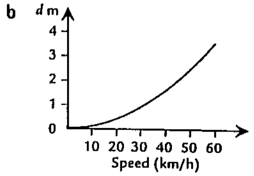
2 a b c d e f



3 a

s (km/h)	0	10	20	30	40	50	60
d (m)	0	0.1	0.4	0.9	1.6	2.5	3.6

b



4 a growth: B, D, F; decay: A, C, E
b linear: A, B; exponential: C, F; quadratic: D, E

5 Growth curves slope upwards, decay curves slope downwards. Linear means straight lines ($y = mx + c$), with constant increases or decreases. Exponentials are curves ($y = a \times b^x$) with constant percentage increases or decreases. Quadratics are curves ($y = x^2$ or $y = k - x^2$) with increasing increases or decreases.

6 a linear b quadratic c linear
d exponential e quadratic f exponential

7 a $y = 2x - 1$ b $y = 9 - x^2$
c $y = 9 - 3x$ d $y = 4 \times 2^{-x}$
e $y = (x + 1)^2$ f $y = 2 \times 2^x$

8 a $x = 0$ or 2 b $x = 4$ or −2
c $x = 3$ or −1 d $x = 1$

9 a $x^2 + 7x + 10$ b $x^2 - x - 2$
c $6x^2 - 11x + 4$ d $9x^2 - 12x + 4$
e $x^2 - 9$ f $4x^2 - 11x + 6$

10 a $(x + 1)(x + 3)$ b $(x - 1)(x - 3)$
c $(x - 6)(x + 2)$ d $(x - 2)^2$
e $(x - 20)(x + 20)$ f $(x - 3)(x + 7)$

11 a $x = 2\frac{1}{2}$ or −3 b $x = 0$ or 1
c $x = 2$ or −1 d $x = 4$ or −3
e $x = 5$ or −2 f $x = 3$ or 6
g $x = -9$ h $x = 3$ or 10
i $x = -2$ or −25

12 a $x = \pm\sqrt{5}$ b $x = 2$
c $x = 1$ or −5 d $x = \pm\frac{\sqrt{14}}{2}$

13 Equation b has no solutions.
a $x = 0.73$ or −2.73
c $x = 2.73$ or −0.73

14 4 m, 7.5 m, 8.5 m

15 a $y = 5 - (x - 2)^2$ or $y = -x^2 + 4x + 1$
b $y = x^2 - 9$ c $y = x^2 - 2x + 2$

16 a 0 b $\frac{2\sqrt{6}}{3}$ c $\frac{\sqrt{10}}{2}$
d $7 - 2\sqrt{10}$ e 24 f 18

Quadratics 2

1 a $(x + 1)^2 + 4, x = -1, 4$
b $(x - 1)^2 - 6, x = 1, -6$
c $(x - 1)^2 - 2, x = 1, -2$
d $(x + 1)^2 + 0, x = -1, 0$
e $(x + 2)^2 + 1, x = -2, 1$
f $(x - 2)^2 - 9, x = 2, -9$
g $(x - 2)^2 - 5, x = 2, -5$
h $(x + 2)^2 - 3, x = -2, -3$
i $(x + 3)^2 - 4, x = -3, -4$
j $(x - 3)^2 - 14, x = 3, -14$
k $(x - 3)^2 - 10, x = 3, -10$
l $(x + 3)^2 - 8, x = -3, -8$
m $(x + 4)^2 - 11, x = -4, -11$
n $(x - 4)^2 - 21, x = 4, -21$
o $(x - 4)^2 - 17, x = 4, -17$

2 a y is higher or lower according to its value.
b +, x is negative; −, x is positive.
c x value is opposite sign, half the magnitude.

3 a $x = -3$ or 1 b $x = 1 \pm \sqrt{6}$
c $x = 1 \pm \sqrt{2}$ d $x = -1$
e no solutions f $x = 5$ or −1
g $x = 2 \pm \sqrt{5}$ h $x = -2 \pm \sqrt{3}$
i $x = -1$ or −5 j $x = 3 \pm \sqrt{14}$
k $x = 3 \pm \sqrt{10}$ l $x = -3 \pm 2\sqrt{2}$
m $x = -4 \pm \sqrt{11}$ n $x = 4 \pm \sqrt{21}$
o $x = 4 \pm \sqrt{17}$

4 a $x = -2$ or 1 b $x = 2$ or −1
c $x = -1$ or −2 d $x = 3.6$ or −0.6
e $x = -0.4$ or −4.6 f $x = 5.4$ or −0.4
g $x = 7.1$ or −0.1 h $x = -0.1$ or −6.9
i $x = -2\frac{1}{2}$ or 1 j $x = 3.3$ or −0.8
k $x = -1$ or −2.5 l $x = 5$ or $-\frac{1}{2}$
m $x = -2$ or $\frac{2}{3}$ n $x = 2.1$ or −0.8
o $x = 1.8$ or −0.2

5 a $(x + \frac{1}{2})^2 + 1\frac{3}{4}, x = -\frac{1}{2}, 1\frac{3}{4}$
b $(x - \frac{1}{2})^2 - 2\frac{1}{4}, x = \frac{1}{2}, -2\frac{1}{4}$
c $(x - \frac{1}{2})^2 - 1\frac{1}{4}, x = \frac{1}{2}, -1\frac{1}{4}$
d $(x + \frac{1}{2})^2 + \frac{3}{4}, x = -\frac{1}{2}, \frac{3}{4}$
e $(x + 1\frac{1}{2})^2 - \frac{1}{4}, x = -1\frac{1}{2}, -\frac{1}{4}$
f $(x - 1\frac{1}{2})^2 - 4\frac{1}{4}, x = 1\frac{1}{2}, -4\frac{1}{4}$
g $(x - 1\frac{1}{2})^2 - 3\frac{1}{4}, x = 1\frac{1}{2}, -3\frac{1}{4}$
h $(x + 1\frac{1}{2})^2 - 1\frac{1}{4}, x = -1\frac{1}{2}, -1\frac{1}{4}$
i $(x + 2\frac{1}{2})^2 - 4\frac{1}{4}, x = -2\frac{1}{2}, -4\frac{1}{4}$
j $(x - 2\frac{1}{2})^2 - 8\frac{1}{4}, x = 2\frac{1}{2}, -8\frac{1}{4}$
k $(x - 2\frac{1}{2})^2 - 7\frac{1}{4}, x = 2\frac{1}{2}, -7\frac{1}{4}$
l $(x + 2\frac{1}{2})^2 - 5\frac{1}{4}, x = -2\frac{1}{2}, -5\frac{1}{4}$
m $(x + 3\frac{1}{2})^2 - 10\frac{1}{4}, x = -3\frac{1}{2}, -10\frac{1}{4}$
n $(x - 3\frac{1}{2})^2 - 14\frac{1}{4}, x = 3\frac{1}{2}, -14\frac{1}{4}$
o $(x - 3\frac{1}{2})^2 - 13\frac{1}{4}, x = 3\frac{1}{2}, -13\frac{1}{4}$

6 a no solutions b $x = 2$ or −1
c $x = -1$ or −2 d $x = \pm\sqrt{2}$
e $x = -2\frac{1}{2} \pm \frac{\sqrt{17}}{2}$ f $x = 2\frac{1}{2} \pm \frac{\sqrt{33}}{2}$
g $x = 3\frac{1}{2} \pm \frac{\sqrt{53}}{2}$ h $x = 0$ or −7
i no solutions j $x = \frac{5}{4} \pm \frac{\sqrt{65}}{4}$
k $x = -1$ or $-2\frac{1}{2}$ l $x = \pm\frac{\sqrt{5}}{\sqrt{2}}$ or $x = \pm\frac{\sqrt{10}}{2}$
m no solutions
n $x = \frac{2}{3} \pm \frac{\sqrt{19}}{3}$ o $x = \frac{5}{6} \pm \frac{\sqrt{37}}{6}$

7 a $v = 50$

t	0	1	2	3	4	5
h	0	45	80	105	120	125

t	6	7	8	9	10
h	120	105	80	45	0

$v = 40$

t	0	1	2	3	4	5	6	7	8
h	0	35	60	75	80	75	60	35	0

$v = 30$

t	0	1	2	3	4	5	6
h	0	25	40	45	40	25	0

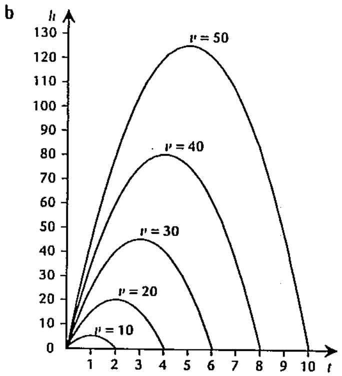
$v = 20$

t	0	1	2	3	4
h	0	15	20	15	0

$v = 10$

t	0	1	2
h	0	5	0

b



c $v = 50$: $t = 5$, 125 m
$v = 40$: $t = 4$, 80 m
$v = 30$: $t = 3$, 45 m
$v = 20$: $t = 2$, 20 m
$v = 10$: $t = 1$, 5 m

d time $= \frac{v}{10}$, maximum height $= \frac{v^2}{20}$
If $h = t(v - 5t) = -5t^2 + vt$
Then $h = -5\left(t^2 - \frac{vt}{5} + \frac{v^2}{100}\right) + 5\frac{v^2}{100}$
$= -5\left(t - \frac{v}{10}\right)^2 + \frac{v^2}{20}$
Thus maximum is $\frac{v^2}{20}$ reached when $t = \frac{v}{10}$

e $v = 50$: $t = 5 \pm \frac{5\sqrt{2}}{2}$; $v = 40$: $t = 4 \pm 2\sqrt{2}$
$v = 30$: $t = 3 \pm \frac{3\sqrt{2}}{2}$; $v = 20$: $t = 2 \pm \sqrt{2}$
$v = 10$: $t = 1 \pm \frac{\sqrt{2}}{2}$

f $t = \frac{v}{10} \pm \frac{v\sqrt{2}}{20}$
Proof: half maximum height $= \frac{v^2}{40}$
Solve: $t(v - 5t) = \frac{v^2}{40}$
leads to $200t^2 - 40vt + v^2 = 0$
Quadratic formula or completing square gives:
$t = \left(\frac{1}{10} \pm \frac{\sqrt{2}}{20}\right)v$ or $\frac{v}{10} \pm \frac{v\sqrt{2}}{20}$

8 c $v = 50$: maximum 62.5 when $x = 125$
$v = 40$: maximum 40 when $x = 80$
$v = 30$: maximum 22.5 when $x = 45$
$v = 20$: maximum 10 when $x = 20$
$v = 10$: maximum 2.5 when $x = 5$

d Maximum is $\frac{v^2}{40}$ when $x = \frac{v^2}{20}$.
$y = -\frac{10x^2}{v^2} + x$
$= -\frac{10}{v^2}\left(x^2 - \frac{v^2}{10} + \frac{v^4}{400}\right) + \frac{v^4}{400} \times \frac{10}{v^2}$
$= -\frac{10}{v^2}\left(x - \frac{v^2}{20}\right)^2 + \frac{v^2}{40}$, hence answer

e 250, 160, 90, 40, 10 for decreasing speeds

f Horizontal is distance between two zeros.
$y = 0$ if $x - \frac{10x^2}{v^2} = 0 \rightarrow x\left(1 - \frac{10x}{v^2}\right) = 0$
$x = 0$ or $x = \frac{v^2}{10}$
$\therefore$ horizontal distance is $\frac{v^2}{10}$.

9 c When speed (x) is zero the ball has stopped rising and is about to start descending, hence it is at its maximum height.
d Because it is a parabola with no x term, it means what happens on the way up will be reflected on the way down.
e 125, 80, 45, 20, 5 for decreasing v
f Maximum occurs when $x = 0$.
Then $y = 0.05(v^2 - x^2) = 0.05v^2$ or $\frac{v^2}{20}$
g If $x < -50$, this is before it was thrown and $x > 50$ would correspond to after it landed (or if it started falling into a hole in the ground!)

Modelling Functions

1 $S = 3L - 25$
2 $h + w = 6$ or $h = 6 - w$ or $w = 6 - h$
3 $A = h(6 - h)$ 4 $w = 20d$
5 $m = 2^c - 1$ 6 $n = 200 \times 0.9^n$
7 $L = 25t^2$

SOLUTIONS 2

Exponential Growth & Decay

1. a 3200 mL   b 5°C
   c $V = 12.5(2)^{\frac{t}{5}}$ d 22°C
2. a 1980, 1987
   b $A = 21(2)^{\frac{n}{7}}$ or $21(1.1)^n$
   c Yes, $1.1^7 \approx 2$ or $2^{\frac{1}{7}} \approx 1.1$
   d 2000
3. a A algae   B wheat
   C bacteria   D duckweed
   E ape   F kitten
   b F (kitten) $m = 1.1^n$ (n = wks, m = mass)
   D (duckweed) $m = 1.5^n$ (n = wks)
   E (ape) $m = 1.2^n$ (n = months)
   C (bacteria) $m = 1.02^n$ (n = minutes)
   A (algae) $m = 1.07^n$ (n = hours)
   B (wheat) $m = 1.05^n$ (n = days)
   d 35, 14, 10, 7.3, 3.8, 1.7
   e $g = \frac{70}{t}$ or $gt = 70$
4. a 1.789, $c = 4(1.789)^{\frac{n}{5}}$
   b 1.23%   c 12.3%
   d

Year	1971	1976	1981	1986
Formula	7	13	23	41
Real	7	18	24	36

   e \$1.31
5. a $E = 4.8(10)^{1.5r}$ or $E = 4.8 \times 31.6^r$
   b because $31.6 = 10^{1.5}$
   c 1000 times as much energy released
   d 1.2×10^9 e 4.8×10^{12} f 4000
6. a $m = 100(0.972)^x$ (one unit = 10 000 years)
   b $m = 100(0.797)^x$ (units are minutes)
   c $m = 100(0.587)^x$ (units are minutes)
   d $m = 100(0.544)^x$ (units are minutes)
7. a error (undefined)   b −1 on graph calc
   c error (undefined)   d +1 on graph calc
9. a $P = 0.9^x$ or $P = 100(0.9)^x$
   b $V = 0.9^x$
   c $P = P_0(0.96)^x$ (P_0 is original pollution at factory)
   d $t = T(0.8)^{\frac{h}{5}}$ (T is surface temperature)

Guitar Frets

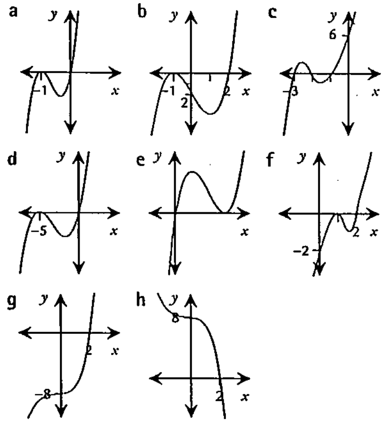
1 frequencies: 123.471, 138.591, 146.832, 164.814, 184.997, 220
2 4:5:6
5 0.945 average in this table
6 Yes
7 $0.5^{\frac{1}{12}} = 0.944$; very close

Cubic Functions

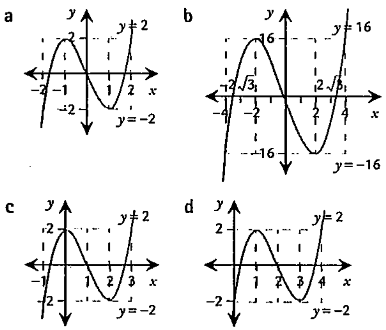
1	a D	b C	c B	d A
2	a C	b B	c A	d D
3	a D	b A	c B	d C
4	a B	b D	c C	d A
5	a D	b A	c C	d B
6	a A	b C	c D	d B

7 a $y = x(x-5)^2$
   b $y = (x-2)(x+1)(x+4)$
   c $y = (x+3)^3 + 2$
   d $y = (x+2)^2(x+1)$

8



9



Maximum Volume

1 2 cm   2 20 cm   3 800 cm³
4 3 cm, 18 cm square, 972 cm³, larger
5 5 cm, 14 cm square, 980 cm³, larger
6

Square	1	2	3	4	5
Height	1	2	3	4	5
Length	22	20	18	16	14
Width	22	20	18	16	14
Volume	484	800	972	1024	980

Square	6	7	8	9	10
Height	6	7	8	9	10
Length	12	10	8	6	4
Width	12	10	8	6	4
Volume	864	700	512	324	160

7 1024 cm³   8 4 cm
9 You'd have to try fractions, like 4.1 cm or 3.9 cm.
10 $24 - 2x$
13 Volume is zero because the base is a square of side length zero.
14 $x = 4$
15 a 1 cm   b 2 cm   c 3 cm   d 5 cm
16 a 60 cm × 60 cm × 20 cm
   b 80 cm × 80 cm × $26\frac{2}{3}$ cm
   c 72 cm × 72 cm × 24 cm
   d 140 cm × 140 cm × $46\frac{2}{3}$ cm