

Year 10 Mathematics Development

Investigation
“Lines on a Plane”

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1. Aim

In investigating, my objective was that when finished, is given any possible plane, I would have the correct formula to find out (from the number of lines and a list of special features) the bounded, unbounded and total number of regions, and the number of intersections in that plane. This aim became more pronounced and refined the further I took the investigation.

The purpose of this report is to summarise and conclude these investigations.

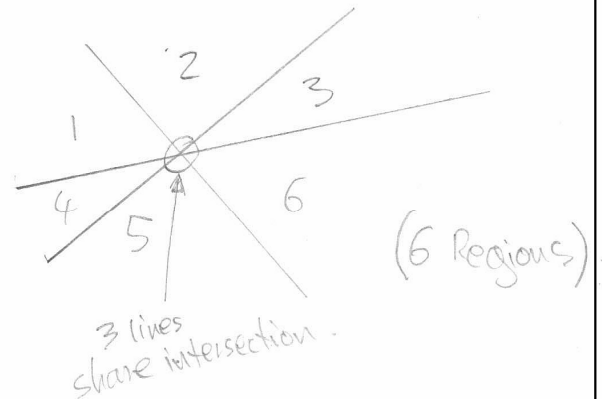
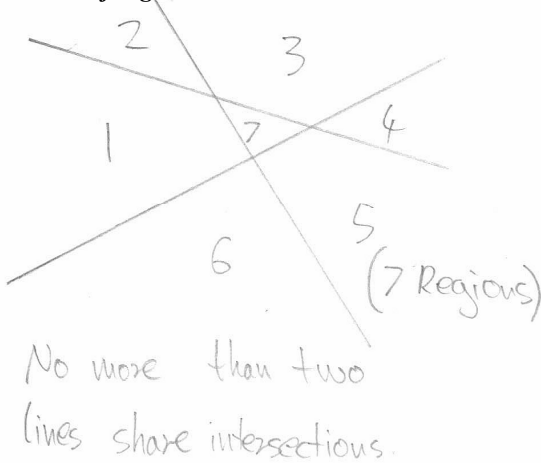
2. First Steps

My first steps were investigating the bounded and unbounded regions, and number of intersections, and attempting to relate these values to the number of lines in the plane.

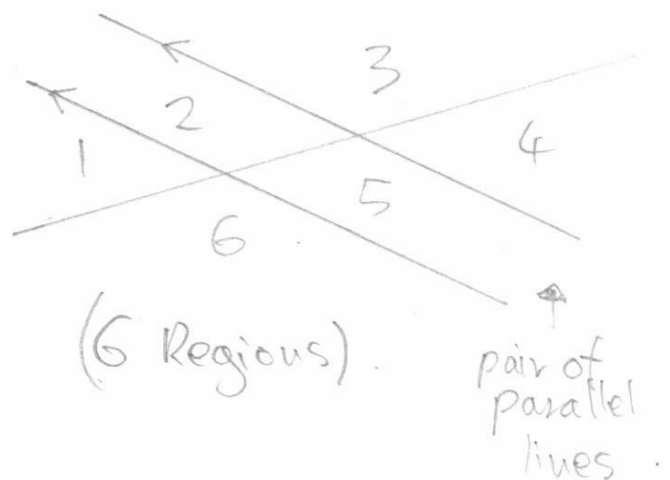
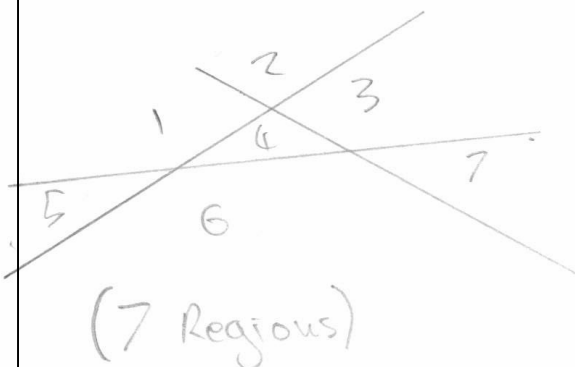
2.1 Variables and Constraints

I soon realised that there were a number of variables and it would be impossible to investigate them all at once.

1. An intersection with more than two lines would change the number of regions

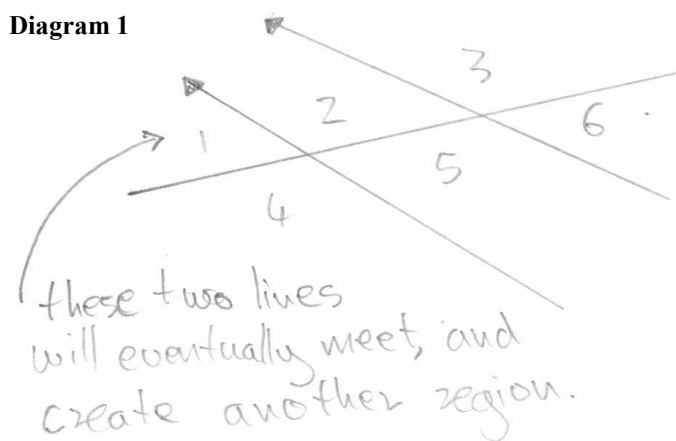
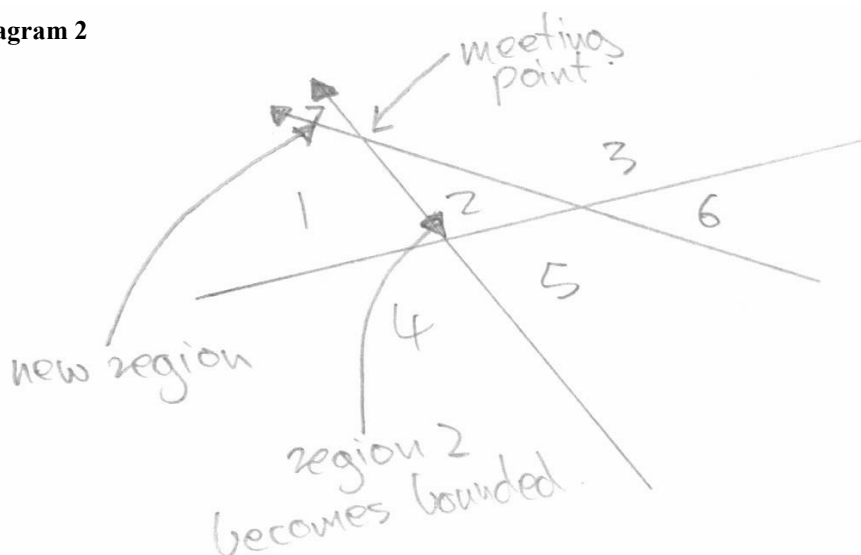


2. Parallel lines would affect the number of regions



In order to experiment with the number of regions and intersections I needed to constrain these variables so they did not interfere with the results.

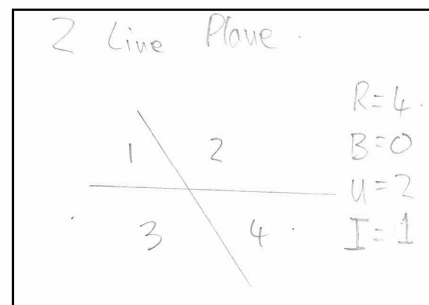
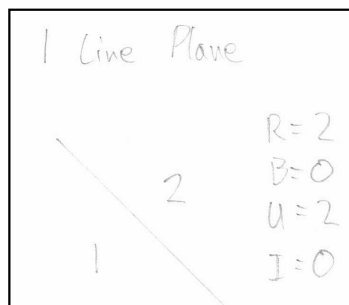
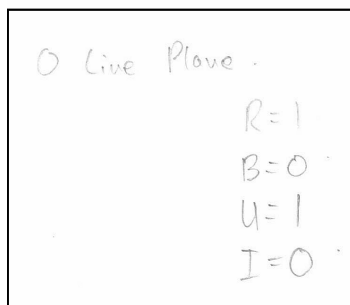
Another catch that got in the way was the fact that the lines on the plane are infinite and diagrams are not. This means that lines that are inclined towards each other (diagram 1) will eventually intersect, but not in the diagram, the result being that a region is not counted (diagram 2).

Diagram 1**Diagram 2**

For this reason I had to sharpen the angles of some of the lines so they would meet in the limited space.

2.2 Process

I started my investigations with a zero line plane, then a one, two, three, and so on up to seven lines.



I drew a number of different variations of each type to check that the results remained the same.

2.3 Observations

The first relationship I noticed (even before drawing up a results table) was that the number of intersections in plane (L) was equal to the number of bounded regions in plane ($L+1$).

Another observation was that (I) was increasing by 1, then 2, then 3, then 4... with every increase of (L). This was (although I did not notice it until after I had already formulated the rule) a sure sign of a second common difference.

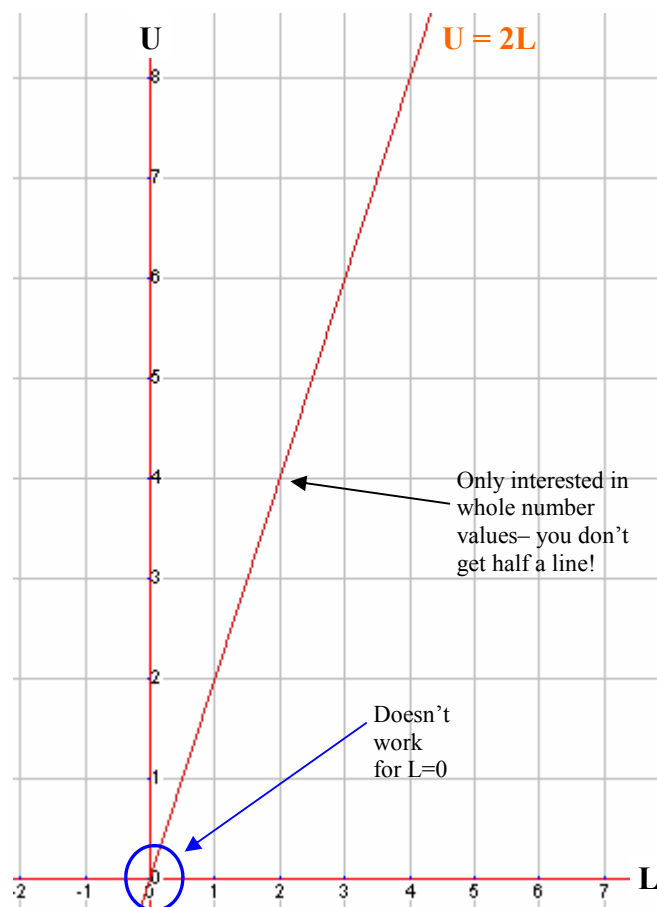
2.4 Results

Lines (L)	0	1	2	3	4	5	6	7
Bounded Regions (B)	0	0	0	1	3	6	10	15
Regions (R)	1	2	4	7	11	16	22	29
Intersections (I)	0	0	1	3	6	10	15	21
Unbounded Regions (U)	1	2	4	6	8	10	12	14
Lines ² (L^2)	0	1	4	9	16	25	36	49

Although my original table did not include them, I felt it was useful to add the unbounded regions ($R-B$) and L^2 (due to the hint that quadratics might come out of the investigation).

2.5 Drawing Rules

The first (and most obvious) relationship is between the lines and unbounded regions; $U = 2L$. This is maintained for all the values present except ($L=0$).



And I soon found another:

$$I = [L \times (L - 1)] / 2$$

...which can be simplified to: $I = (L^2 - L) / 2$ or $I = 0.5L^2 - 0.5L$

This was the key to all the relationships; (I) comes into all of the formulae and using (I) it's easy to work out the others. For example:

$$B = I - (L - 1)$$

$$R = L + I + 1$$

I realised that by substituting the formula for (I) into each of these equations I could give B and R their own individual equation, and make them non-reliant on knowing the value of (I).

I started with (R):

$$R = I + L + 1$$

$$= (L^2 - L) / 2 + L + 1$$

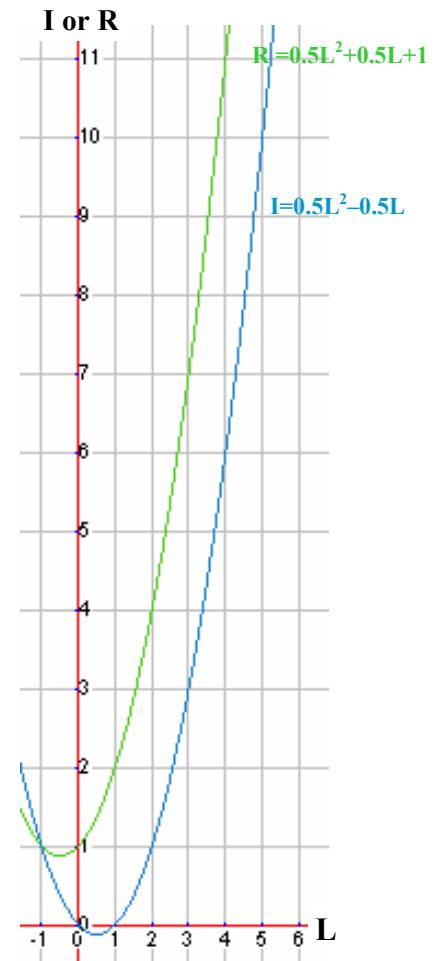
$$2R = L^2 - L + 2(L + 1)$$

$$= L^2 - L + 2L + 2$$

$$= L^2 + L + 2$$

$$R = (L^2 + L + 2) / 2$$

$$= 0.5L^2 + 0.5L + 1$$



And followed the same process

for (B):

$$B = I - (L - 1)$$

$$= (L^2 - L) / 2 - (L - 1)$$

$$2B = L^2 - L - 2(L - 1)$$

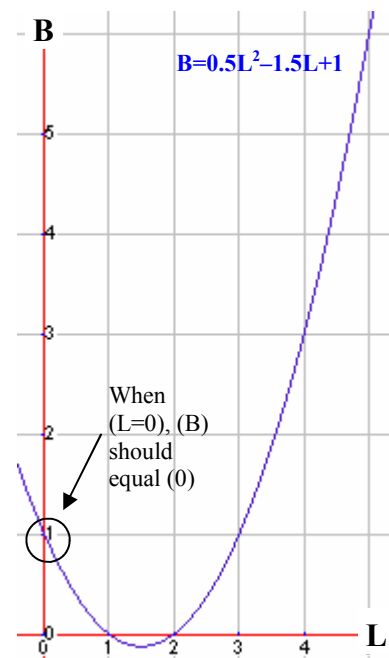
$$= L^2 - L - 2L + 2$$

$$= L^2 - 3L + 2$$

$$B = (L^2 - 3L + 2) / 2$$

$$= 0.5L^2 - 1.5L + 1$$

This rule works for every value of (L) except (once again) the annoying case of (L=0). It gives the value of (B=1), whereas (B) actually equals (0) for this case.



2.6 Workings of the (R) Formula

Looking at the diagrams we can see that:

$$R(L+1) = R(L) + L + 1$$

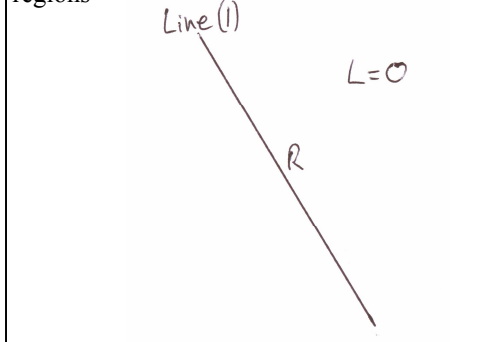
(OR)

Regions of the plane (L+1) = regions of (L) + (L+1)

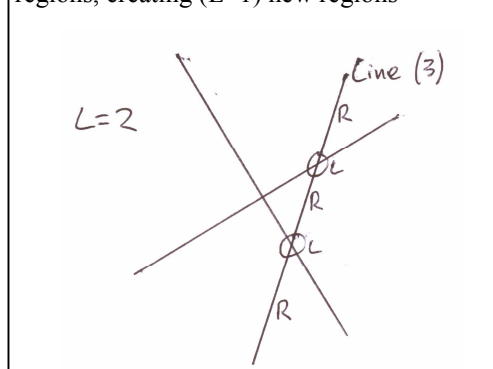
(OR)

Regions of the next plane = regions of this plane, plus the lines of this plane, plus one

Line (1) passes through (L) Lines and (L+1) regions

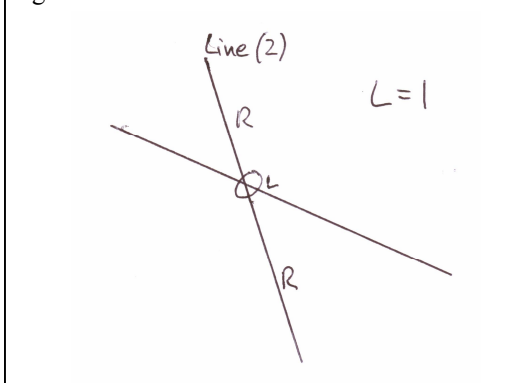


Line (3) passes through (L) Lines and (L+1) regions, creating (L+1) new regions

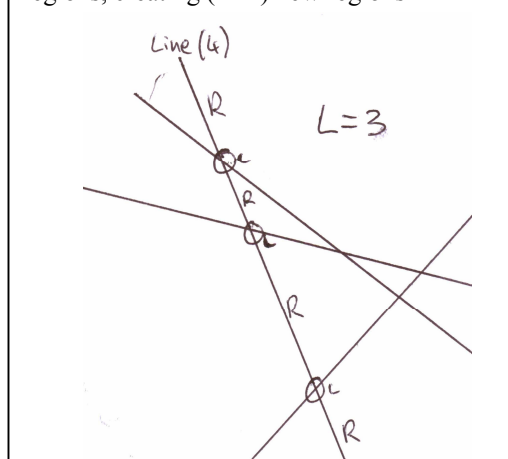


Line (2) passes through (L) Lines and (L+1) regions

Eg. Each region line (2) passes through it splits into two. Line (2) creates two, or (L+1) new regions.



Line (4) passes through (L) Lines and (L+1) regions, creating (L+1) new regions



2.7 Proof

To try and prove at least one of my formulas, I researched methods on the internet. What came up was Mathematical Induction. I have attempted to prove my formula using this method.

(L+1) new regions
are created each time

Lines (L)	0	1	2	3
Regions (R)	1	2	4	7

The formula works for the first case of (L=0)

The above diagrams show that:

$$R(L+1) = R(L) + L + 1$$

The formula for R (L) is:

$$R(L) = 0.5L^2 + 0.5L + 1$$

This means the formula for (L+1) must be:

$$\begin{aligned} R(L+1) &= 0.5(L+1)^2 + 0.5(L+1) + 1 \\ &= 0.5(L+1)(L+1) + 0.5(L+1) + 1 \\ &= 0.5(L^2 + L + L + 1) + 0.5(L+1) + 1 \\ &= 0.5L^2 + 0.5L + 0.5L + 0.5 + 0.5L + 0.5 + 1 \\ &= (0.5L^2 + 0.5L + 1) + L + 1 \end{aligned}$$

And this shows that:

$$R(L+1) = R(L) + L + 1$$

(i.e.) if the formula works for (L) it also works for (L+1)

We know the formula works for (L=0) (the first case), and that if it works for (L) it will work for (L+1). As it works for L=0, we know it will work L=1 (the next case), and if we know it works for L=1, then it will work for L=2, and if we know it works for L=2, then it will for (L=3)... and so on. In this way, we know the formula will continue to work indefinitely.

3. Intersections

Having found some basic rules for the calculation of regions and intersections I decided to experiment with the constraints I had previously imposed on the investigation.

The objective of this section was to assess the change in (R) and (I) as a result of more than two lines sharing an intersection, and try and draw some rules and relationships. I must admit that I was not sure that any would be present.

3.1 Results Table

Lines sharing intersection (N)	2	3	4	5	6	7
Change in regions (C)	0	-1	-3	-6	-10	-15

3.2 Observations

Soon after completing this table, I decided I needed to investigate not only the change in regions but also the change in bounded and unbounded regions. Thus, I checked back over and added (B) into the result lists for each of my diagrams. Offcourse, the final table (comparing lines sharing (I) with change in (B)) was exactly the same as the table above.

This means that the change in regions that occurs when more than two lines share an intersection is entirely a change in bounded regions. The result of understanding this was another interesting revelation, which I had noticed prior but not quite comprehended.

If you look at the results table in 2.4 (particularly the row recording bounded regions) you will notice that each value, when multiplied by negative one (-1), equals exactly its corresponding value in the "Change of regions row" in results table 3.1 (above). The rows are repeated below:

Lines (L)	2	3	4	5	6	7
Change in regions (C)	0	-1	-3	-6	-10	-15
Bounded Regions (B)	0	1	3	6	10	15

This makes perfect sense. Since the change in regions when intersections are shared is entirely in bounded regions, then the change in regions will equal (-B).
(i.e.) $C = -B$

3.3 Drawing Rules

That last observation made drawing rules an incredibly easy process. I just had to use the formula for (B) and ensure the result was multiplied by negative one.

$$B = (L^2 - 3L + 2)/2$$

$$C = (L^2 - 3L + 2)/-2$$

$$= -0.5N^2 + 1.5N - 1$$

There are only two imperfections I can spot with this formula. However, as the points in question ($[N=1]$ and $[N=0]$) are irrelevant to the context of the plane (i.e. if zero or one lines are present there is no intersection, and thus, no lines can be sharing an intersection) the questionable values are hardly surprising. This is especially the case as we know there was already an imperfection with the original formula for calculating bounded regions.

3.4 The effect on (I)

It did not take me long to realise that there was one factor I needed to look at further; the effect on (I) itself (the number of intersections) when more than two lines shared an intersection.

3.4.1 Table of Results

Lines sharing intersection (N)	2	3	4	5	6	7
Change in no. of intersections (K)	0	-2	-5	-9	-14	-20

3.4.2 Rules

In this case, the easiest way to find a formula would be to use the common second difference. But before I could do this I had to find the value of (K) when ($N=0$) and ($N=1$), and experimented with a series of diagrams to find these. Below is the table I came up with.

Lines sharing intersection (N)	0	1	2	3	4	5
Change in no. of intersections (K)	1 (c)	1	0	-2	-5	-9
First difference	0 (a+b)		-1	-2	-3	-4
Common second diff	-1 (2a)		-1	-1	-1	

Thus: $c=1$, $a=-0.5$, $b=0.5$

$$K = aN^2 + bN + c$$

$$= -0.5N^2 + 0.5N + 1$$

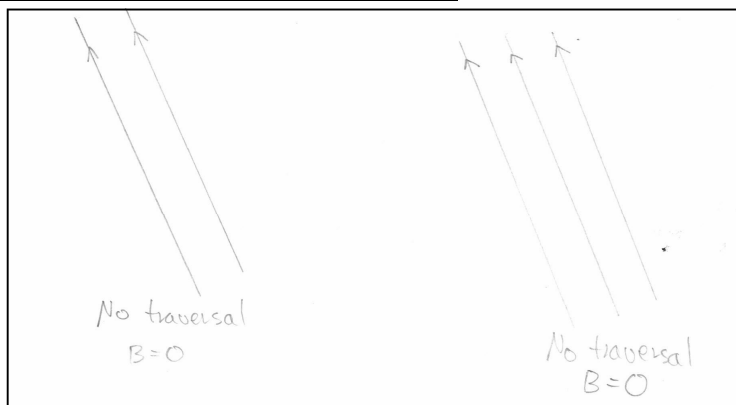
4. Parallel Lines

In this section I investigate the second variable of parallel lines and their effect while disallowing any intersections to be shared by more than two lines.

4.1 Results

No. of Parallel Lines in each individual set (Y)	2	3	4	5
Change in no. of intersections and regions per set (P)	-1	-3	-6	-10

NOTE: The change in change in number of regions is completely in bounded regions if there is at least one traversal of each set of parallel lines, whether this traversal be another set of parallel lines or simply a single line. In the case of parallel lines with no traversal, bounded regions always equal zero (no traversal... $B=0$).

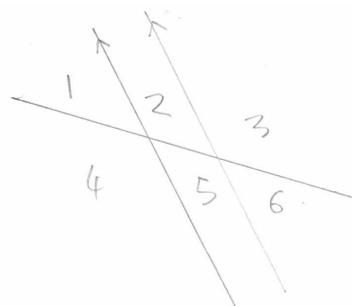


If there is only one traversal of a single set of parallel lines (B) also equals (0).

In this case $R = R(N) - B(N)$

That is, the change in number of regions is entirely in bounded regions.

(N) Represents a normal/conventional plane with no special features



$$B=0$$

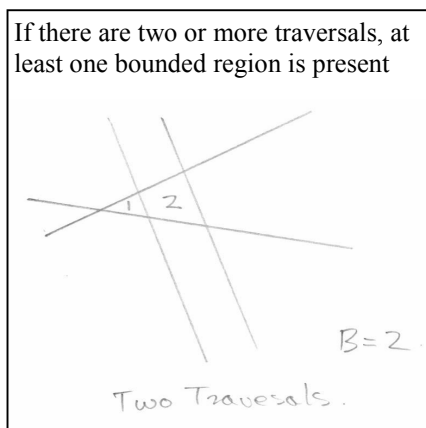
$$R=6$$

$$R = R(N) - B(N)$$

$$= 7 - 1$$

$$= 6$$

If there are two or more traversals, at least one bounded region is present



$$B=2$$

Two Traversals.

4.2 Rules

To find a formula for these values I used the method of common second difference. In doing this I had to input the values for ($Y=0$) and ($Y=1$) (these are: $P=0$ and $P=0$). It made no sense to put these into the table above as the very meaning of parallel lines requires two or more lines to be in the set. There is no such thing as parallel lines with one or zero lines in the set- or not in practise anyway, perhaps theoretically.

Via common second diff. we find:

$$c=0, a=-0.5, b=0.5$$

And thus...

$$P = aY^2 + bY + c$$

$$= -0.5Y^2 + 0.5Y$$

5. Using in Practise

Now we know all these rules let's have a look at how we can use them, and see if the objective has been achieved.

5.1 Process

Outlined below is a simple three step process that can be used in practise.

5.1.1 Step 1

Always the first step is to determine the bounded regions, total regions and intersections of a conventional form of the given plane. That is, disregard for the moment the variables of parallel lines and intersections. Simply take the number of lines (L) and from it, calculate (B), (R) and (I).

Lines= 11

(N) indicates that these would be the values for a normal/conventional plane

$$\begin{aligned} R(N) &= (L^2 + L + 2)/2 \\ &= (11^2 + 11 + 2)/2 \\ &= 134/2 \\ &= 67 \end{aligned}$$

(It is sometimes useful to calculate (U) and (R) and from that (B) as a shortcut. I felt [if not using a calculator] it was a good idea to do it both ways, and in doing so, check the answer.)

$$\begin{aligned} U(N) &= 2L \\ &= 2 \times 11 \\ &= 22 \end{aligned}$$

$$\begin{aligned} B(N) &= R - U \\ &= 67 - 22 \\ &= 45 \end{aligned}$$

(Also, it is sometimes easier to use the whole number form of the equation (below) rather than messing with decimals)

$$\begin{aligned} B(N) &= (L^2 - 3L + 2)/2 \\ &= (11^2 - 3 \times 11 + 2)/2 \\ &= (121 - 33 + 2)/2 \\ &= 90/2 \\ &= 45 \end{aligned}$$

$$\begin{aligned} I(N) &= (L^2 - L)/2 \\ &= (11^2 - 11)/2 \\ &= 110/2 \\ &= 55 \end{aligned}$$

5.1.2 Step 2

The second step is to look at any special features that might be present and calculate the changes that will occur to (I), (R) and (B) as a result of these.

The best way is to set up a table like below and then [calculate the necessary changes](#).

Note again, that the change in (B) will always be the same as the change in (R) so long as there is a traversal of the parallel lines. If there is not a traversal of the parallel lines ($B=0$).

Special Features	Resultant change in (R)	Resultant change in (B)	Resultant change in (I)
1x pair of parallel lines	-1	-1	-1
1x triplet parallel lines	-3	-3	-3
2x 3-line-share intersection	$2 \times -1 = -2$	$2 \times -1 = -2$	$2 \times -2 = -4$
TOTAL =	-6	-6	-8

Calculations:

C = Change in (R) and (B) due to lines sharing (I) [per intersection shared]

$$C = (L^2 - 3L + 2)/-2 \\ = -1$$

K = Change in (I) due to lines sharing (I) [per intersection shared]

$$K = -0.5N^2 + 0.5N + 1 \\ = -2$$

P = Change in (R), (I) and (B) due to a set of parallel lines

Y = Number of parallel lines in set

$$Y=2$$

$$P = -0.5Y^2 + 0.5Y \\ = -1$$

$$Y=3$$

$$P = -0.5Y^2 + 0.5Y \\ = -3$$

5.1.3 Step 3

The third step is the simplest of the lot. All that is needed is to make the adjustments to (R), (B), and (I).

$$R = 67-6 \\ = 61$$

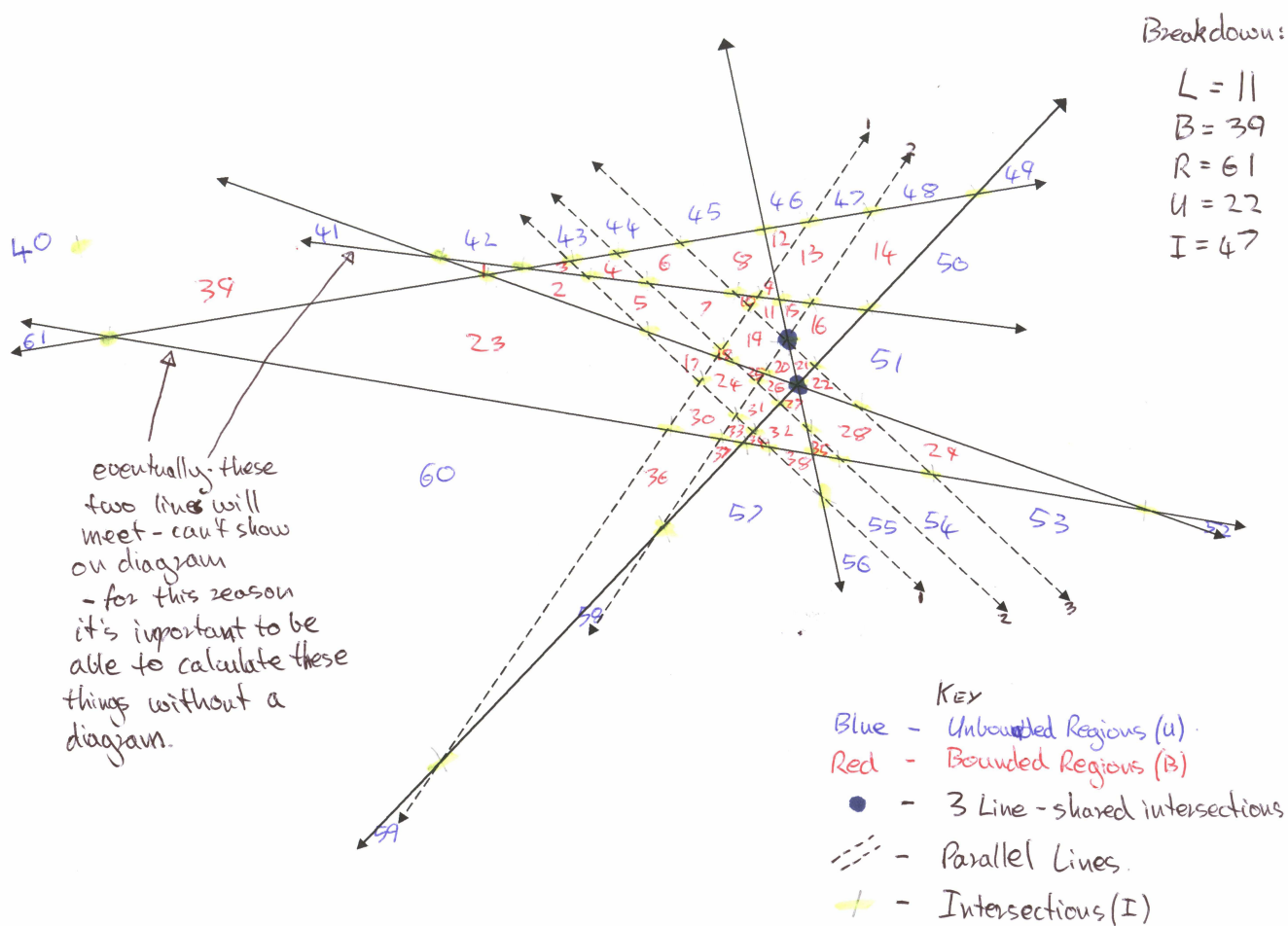
$$B = 45-6 \\ = 39$$

$$I = 55-8 \\ = 47$$

Thus, the plane has 61 regions, 39 bounded regions, 22 unbounded regions (i.e. $U=R-B$), and 47 intersections.

5.2 Model

Below is a model of the plane described in (5.1).



6. Summary

6.1 Complete Table of Formula

6.1.1 Formula for conventional plane

Calculates:	Formula	Notes
Bounded Regions (B)	$B = (L^2 - 3L + 2)/2$ $= 0.5L^2 - 1.5L + 1$	Doesn't work for (L=0). When (L=0), (B) should equal (0).
Regions (R)	$R = (L^2 + L + 2)/2$ $= 0.5L^2 + 0.5L + 1$	
Intersections (I)	$I = (L^2 - L) / 2$ $= 0.5L^2 - 0.5L$	
Unbounded Regions (U)	$U = 2L$	Doesn't work for (L=0). When (L=0), (U) should equal (1)

6.1.2 Formula for calculating Special Features

Calculates:	Formula	Notes
Change (C) in regions and bounded regions due to (N) lines sharing an intersection (per intersection)	$C = (L^2 - 3L + 2)/-2$ $= -0.5N^2 + 1.5N - 1$	Disregard answers for (N=0) and (N=1)
Change (K) in number of intersections as a result of (N) lines sharing an intersection (per intersection)	$K = -0.5N^2 + 0.5N + 1$	
Change (P) in number of intersections and regions (<i>*including bounded</i>) as a result of (Y) parallel lines (per set of (Y) lines)	$P = -0.5Y^2 + 0.5Y$	<i>* only if there is a traversal of the parallel lines- otherwise, (B=0)</i>

6.2 Conclusion

I believe that with the formulae outlined in this summary, I could, when given the number of lines on any plane and any special features present, accurately calculate the number of intersections and regions (total, bounded and unbounded) in that plane (without drawing it). Thus, I have achieved my objective outlined in (1.0), and leave this investigation satisfied with my results. The formulae may not be perfect in all cases but they do provide an excellent tool in investigating the astounding geometry and interrelationships of the plane.