ST STEPHENS SCHOOL, CARRAMAR

YEAR 12: DISCRETE MATHEMATICS, SEMESTER 2 2007

TEST : PROBLEM SOLVING

Name: ______________________ **Total:** ______ / 62

Teacher: ______________________

Time: 70 mins **1 Page of notes (2 sided). Use calculator where necessary.**

GOOD LUCK!

1.(10 marks)

Find the next 3 numbers in each sequence:

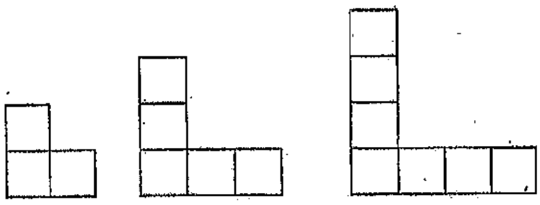
a) 8, 5, 2, -1, ..-4.., ..-7..., ..-10.. ✓
$-3 \quad -3 \quad -3$

b) 2, 4, 7, 11, ..16.., ..22.., ..29... ✓
$+2 \quad +3 \quad +4 \quad +5 \quad +6 \quad +7$

c) 2, 6, 18, 54, ..162, ..486, .1458 ✓
$\times 3 \quad \times 3 \quad \times 3$

d) 9, 6, 4, 3, ..3..., ..4..., ..6.... ✓
$-3 \quad -2 \quad -1 \quad +0 \quad +1 \quad +2$

e) investigate the following L shapes and the number of squares required to make them.



1 2 3

Fill in the table:

L Shape(L)	0	1	2	3	4	5	6	7	8	9	10
Number of Squares(N)	1	3	5	7	9	11	13	15	17	19	21

✓✓

f) What is the rule that relates the number of squares to the number of the L shape?

① ② simplified ③ correct variables (N & L)

$N = 2L + 1$ ✓✓✓

g) Use the rule to find the number of squares in the 300th L shape

$N = 2 \times 300 + 1$

$= 601$

601 squares ✓

2. (14 marks)

The Fibonacci sequence has the recursive rule:

$$F_{n+2} = F_{n+1} + F_n \quad , \quad F_1 = F_2 = 1$$

(a) List the first 5 terms of the Fibonacci sequence. [2]

1, 1, 2, 3, 5.

(b) Find n in each of the following:

(i) $F_1 + F_2 + F_3 = F_n - 1$ [2]

$1 + 1 + 2 = F_n - 1$

$4 = F_n - 1$

$F_n = 5 \therefore \quad n = 5$

(ii) $F_1 + F_2 + F_3 + F_4 = F_n - 1$ [2]

$1 + 1 + 2 + 3 = F_n - 1$

$F_n = 8$

$n = 6$

(iii) $F_1 + F_2 + F_3 + F_4 + F_5 + \ldots\ldots F_{50} = F_n - 1$ [2]

$n = 52$

(iv) $F_1 + F_2 + F_3 + F_4 + F_5 + \ldots\ldots F_{99} + F_{100} = F_n - 1$ [2]

$n = 102$

(c) The Lucas sequence has the recursive rule:

$L_{n+2} = L_{n+1} + L_n \quad , \quad L_1 = 1 \quad L_2 = 3$

1, 3, 4, 7, 11, 18,

By considering simple cases and looking for patterns, suggest the values of the constants k and c in the following:

$L_1 + L_2 + L_3 + L_4 + L_5 + \ldots\ldots L_{n-1} + L_n = L_{n+k} - c$ [4]

$L_1 + L_2 + L_3 = L_5 - 3$

$1 + 3 + 4 = 8 = L_5 - 3$

$L_1 + L_2 + L_3 + L_4 = L_6 - 3$

$1 + 3 + 4 + 7 = 15 = L_6 - 3$

$L_1 + L_2 + \ldots\ldots L_n = L_{n+2} - 3$

$\therefore \quad k = 2$

$c = 3$

3. (23 marks)

The following rules of divisibility apply to counting numbers.

A number is divisible by

2, if the last digit is divisible by 2.

3, if the sum of its digits is divisible by 3.

5, if the last digit is 5 or 0.

9, if the sum of its digits is divisible by 9.

10, if the last digit is 0.

eg. If we were testing a number, say 48, for divisibility, we would proceed in the way described below:

48 is divisible by 2, since 8 is divisible by 2

48 is divisible by 3, since $4 + 8 = 12$ is divisible by 3

48 is not divisible by 5, since 8 is not 5 or 0

48 is not divisible by 9, since $4 + 8 = 12$ is not divisible by 9

48 is not divisible by 10, since 8 is not 0

(a) By using the word "AND" combine some of the above rules to produce a rule for divisibility by

(i) 6 [2]

If the last digit is divisible by 2
AND
if the sum of its digits is divisible by 3

(ii) 15 [2]

if the sum of its digits is divisible by 3
AND
the last digit is 5 or 0

(iii) 30 [2]

if the sum of its digits is divisible by 3
AND
the last digit is zero.

If the Universal set includes all integers from 1 to 20 inclusive,

(b) List (i) A, which is the set of numbers divisible by 3 [1]

$\{3, 6, 9, 12, 15, 18\}$

(ii) B, which is the set of numbers divisible by 5 [1]

$\{5, 10, 15, 20\}$

(iii) C, which is the set of numbers divisible by 15 [1]

$\{15\}$

(iv) P, which is the set of Prime numbers [2]

$\{2, 3, 5, 7, 11, 13, 17, 19\}$

(iv) F, which is the set of Fibonacci numbers [1]

$\{1, 2, 3, 5, 8, 13\}$

(v) $F \cap A$ [1]

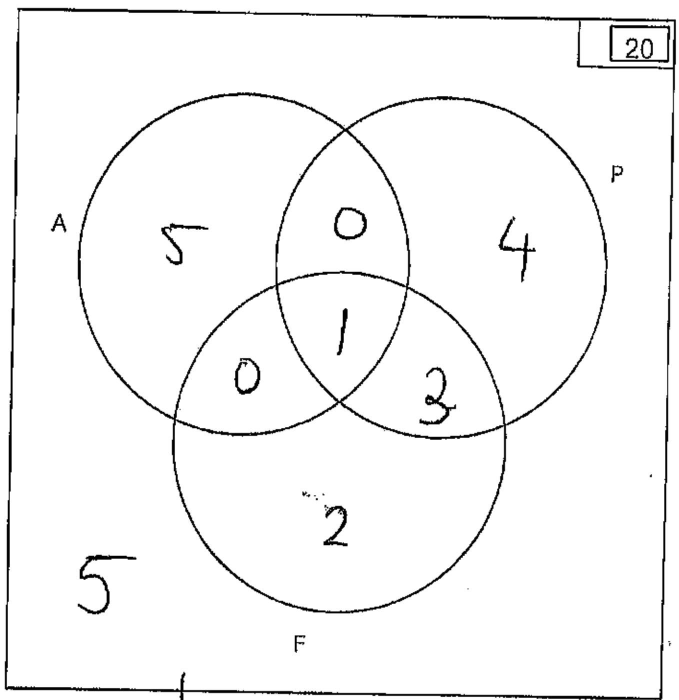
$\{3\}$

(vi) $F \cup A$ [2]

$\{1, 2, 3, 5, 6, 8, 9, 12, 13, 15, 18\}$

9

(c) If A,P and F are as defined earlier, complete the Venn diagram, indicating the ***number of elements*** in each region.(the table below will help you) [8]



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A			✓			✓			✓			✓			✓			✓		
P		✓	✓		✓		✓				✓		✓				✓		✓	
F	✓	✓	✓		✓			✓					✓							

4. [10 marks]

A law of indices states that $a^m \cdot a^n = a^{m+n}$
Example: $3^{-5} \cdot 3^{-3} = 3^{-8}$, or extended to $3^{-5} \cdot 3^{-3} \cdot 3^{-1} = 3^{-9}$

The following sequence shows four factors as the beginning of a sequence of factors.

$3^{-5} \cdot 3^{-3} \cdot 3^{-1} \cdot 3^{1} \cdot 3^{3} \ldots 3^{5} \quad 3^{7} \quad 3^{9} \quad 3^{11} \quad 3^{13}$

6 factors = 1

(a) Evaluate the product of eight such factors. [2]

$3^7 \times 3^9 = 3^{16} = 43,046,721$

(b) Evaluate the product of ten such factors. [2]

$3^{7+9+11+13} = 3^{40} = 1.2158 \times 10^{19}$

(c) Determine the least number of factors of the above sequence that are required to make a product of at least:

(i)3^7 7 factors [2]

(ii)1 6 factors [2]

10 million (ii)10^7 $3^7 = 2187$, $3^{16} = 43,046,721$ 8 factors. [2]

7 factors ✗ 8 factors

5. [5 marks]

Melanie wins Lotteryo.

She is given the choice of:

A One million dollars.

OR

B 1 cent on November 1st, 2 cents on November 2nd, 4 cents on November 3rd, doubling each day until and including, November 30th.

Which prize is more valuable, and by how much?
Show all working. [5]

B $a = 1$ cent
$r = 2$

$$S_{30} = \frac{a(r^n - 1)}{r-1} = \frac{1(2^{30}-1)}{1} = 2^{30} - 1 = 1073741823c$$

B = $10,737,418.23

B is more valuable by $9737418.23

(T_{30} note
5368709)

YEAR 12 DISCRETE MATHEMATICS
SEMESTER 2 2008

PROBLEM SOLVING TEST #2

Name: Solutions Teacher: DA / JB / CS

Time: 55 minutes

Calculators & one A4 page (two sides) of notes PERMITTED

Show **all appropriate working out** for full marks.

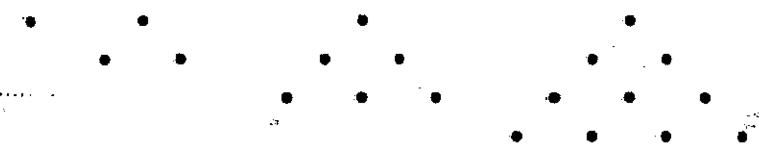
Your mark: ______ /50

__________ %

Parent Signature: ____________________

1 (4 marks)

The diagrams show the first four triangular numbers: 1, 3, 6, 10,
It can be seen that $T_2 = T_1 + 2$, $T_3 = T_2 + 3$, $T_4 = T_3 + 4$.



T_1 T_2 T_3 T_4

1 3 6 10

-1 NO T_1

(a) Generalise from this to write a recursive definition that will define the sequence of triangular numbers. [2]

$T_n = T_{n-1} + n$, $T_1 = 1$

$T_{n+1} = T_n + (n+1)$, $T_1 = 1$

$n = 2$ $T_3 = T_2 + 3 = 3 + 3 = 6$

(b) Find an expression for T_n in terms of n [2]

$T_n = \frac{1}{2}n^2 + \frac{1}{2}n$

RB

1 3 6 10

$1^{st} \longrightarrow$ 2 3 4

$2^{nd} \longrightarrow$ 1 1

• $2a = 1$
$a = \frac{1}{2}$

• $3a + b = 2$
$b = \frac{1}{2}$

• $y = 0$, $c = 0$

$T_0 = 0$

2 (6 marks)

A Harmonic Progression (HP) is obtained when each term of an arithmetic sequence (AP) is inverted.

For example:

1, 2, 3, 4, ... (is the AP) $\Rightarrow$ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (is the HP)

$\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots$ (is the AP) $\Rightarrow$ 12, 6, 4, 3, ... (is the HP)

(a) Find the 10[th] term of the HP in which the second and sixth terms are $\frac{1}{2}$ and $\frac{1}{18}$ respectively. [3]

$2, 6, 10, 14, 18$

$AP \longrightarrow T_2 = 2, \quad T_6 = 18$

$4d = 16 \quad \therefore d = 4$

$a = -2, \quad \therefore T_{10} = -2 + 9 \times 4 = 34$

$HP \longrightarrow T_{10} = \frac{1}{34}$

(b) A sequence of numbers is of $\frac{1}{3}, \frac{1}{8}, \frac{1}{13}, \frac{1}{18}, \ldots$

(i) Show that it is an Harmonic Progression. [1]

$AP \longrightarrow 3, 8, 13, 18 \quad \therefore d = 5$

$\therefore \frac{1}{3}, \frac{1}{8}, \frac{1}{13}$ is HP

(ii) Find the sum of the first 8 terms of the arithmetic sequence which is defined by the above harmonic progression. [2]

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_8 = 4(6 + 7 \times 5)$

$= 164$

2 (6 marks)

A Harmonic Progression (HP) is obtained when each term of an arithmetic sequence (AP) is inverted.

For example:

$1, 2, 3, 4, \ldots\ldots$ (is the AP) $\Rightarrow$ $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (is the HP)

$\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \ldots\ldots$ (is the AP) $\Rightarrow$ $12, 6, 4, 3, \ldots\ldots$ (is the HP)

(a) Find the 10^{th} term of the HP in which the second and sixth terms are $\frac{1}{2}$ and $\frac{1}{18}$ respectively. [3]

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(b) A sequence of numbers is of $\frac{1}{3}, \frac{1}{8}, \frac{1}{13}, \frac{1}{18}, \ldots\ldots$

(i) Show that it is an Harmonic Progression. [1]

$AP \longrightarrow 3, 8, 13, 18 \quad \therefore d = 5$

$\therefore \frac{1}{3}, \frac{1}{8}, \frac{1}{13}$ is HP

(ii) Find the sum of the first 8 terms of the arithmetic sequence which is defined by the above harmonic progression. [2]

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_8 = 4(6 + 7 \times 5)$

$= 164$

3 (10 marks)

(a) A number of conjectures are made below. For each of the following conjectures, state if they are true or false. If it is true, give two examples of when it is true. If it is false, give one example of when it is false.

The set of counting numbers is {1, 2, 3, 4, ... }

(i) If n is an odd counting number greater than 1, then $n^2 - 1$ is always a multiple of 8. [2]

$n = 3;\ n^2 - 1 = 8 = \underline{8 \times 1}$ TRUE

* $n = 5;\ n^2 - 1 = 24 = \underline{8 \times 3}$

(ii) The product of two consecutive counting numbers is always a multiple of 4. [2]

$5 \times 6 = 30$ — not a multiple of 4. FALSE

$\frac{30}{4} = 7.5$

(iii) The product of three consecutive counting numbers is always a multiple of 6. [2]

$2 \times 3 \times 4 = 24 = \underline{6 \times 4}$ TRUE

* $9 \times 10 \times 11 = 990 = \underline{6 \times 165}$

Show multiple

(iv) The sum of three consecutive even counting numbers is always a multiple of 6. [2]

$2 + 4 + 6 = 12 = \underline{6 \times 2}$

* $20 + 22 + 24 = 66 = \underline{6 \times 11}$ TRUE

(b) Justify the claim that: the sum of three consecutive counting numbers is always a multiple of 3. [2]

eg $\underline{1 + 2 + 3 = 6 = 3 \times 2}$ (just example $\frac{1}{2}$)

$n + (n+1) + (n+2) = 3n + 3$

$= 3(n+1)$

ie always a multiple of 3.

7

4 (14 marks)

Consider the following sequence:

T_1	1
T_2	110
T_3	1101001

Each line is obtained by writing 1 in the middle and writing the line above on the left of the 1. On the right of the middle 1, the line above is written again with all the zeros as ones and the ones as zeros.

This means that the next line, T_4 will be:

110100110010110

(a) Write the line for T_5 [1]

1101001100101101001011001101001

A new sequence, D_n, represents the number of digits in each line of the T_n sequence. Therefore,

$D_1 = 1$, $D_2 = 3$ and so on.

(b) Give the value of [3]

(i) D_3 7

(ii) D_4 15

(iii) D_5 31

(c) Give a recursive definition for D_n [2]

$D_n = 2D_{n-1} + 1, \quad D_1 = 1$

OR $[D_n = D_{n-1} + 2^{n-1}, D_1 = 1]$

6

Another sequence, W_n, represents the number of ones in each line of the T_n sequence. Therefore

$W_1 = 1$, $W_2 = 2$, and so on.

(d) Give values for [1]

(i) W_3 .... 4 ✓½

(ii) W_4 .... 8 ✓½

and (iii) a general expression for W_n [2]

1, 2, 4, 8
$a = 1$
$r = 2$

$W_n = ar^{n-1}$ ✓

$W_n = 2^{n-1}$ ✓

Another sequence Z_n represents the number of zeros in each line of the T_n sequence.

(e) Therefore find [5]

(i) Z_1 .... 0 ✓½

(ii) Z_2 .... 1 ✓½

(iii) Z_3 .... 3 ✓

(iv) Z_4 .... 7 ✓

(v) Z_n .... $Z_{n-1} + Z_{n-1} + 1$

$Z_n = 2Z_{n-1} + 1, \ Z_1 = 0$ ✓ ✓

$Z_n = 2^{n-1} - 1$

8

5 (16 marks)

If the Universal set includes all integers from 1 to 20 inclusive,

(a) List (i) • A, which is the set of numbers divisible by 3 [1]

$\{3, 6, 9, 12, 15, 18\}$ ✓

(ii) B, which is the set of numbers divisible by 5 [1]

$\{5, 10, 15, 20\}$ ✓

-1 → { }

(iii) C, which is the set of numbers divisible by 15 [1]

$\{15\}$ ✓

(iv) • P, which is the set of Prime numbers [1]

$\{2, 3, 5, 7, 11, 13, 17, 19\}$ ✓

No mark! $\{1, 1, 2, 3$ etc $\}$ *

(iv) • F, which is the set of Fibonacci numbers [1]

$\{1, 2, 3, 5, 8, 13\}$ ✓

(v) $F \cap A$ [1]

$\{3\}$ ✓

-1 each error

(vi) $F \cup A$ [2]

$\{1, 2, 3, 5, 6, 8, 9, 12, 13, 15, 18\}$ ✓✓

(b) If A,P and F are as defined earlier, complete the Venn diagram, indicating the **number of elements** in each region. [8] *