St Stephen's School – Carramar Campus
Discrete Mathematics – Problem Solving Test 1, 2008

Total Marks: 60 Time Allowed: 60 mins

Name: *Solutions* Mark: Barbels

Parent Signature:

Question 1 [2 marks]

Find five consecutive whole numbers which have a product of 1 028 160. Indicate the method used to solve the problem.

$5\sqrt{1028160} = 15.937$ ✓

Try $14 \times 15 \times 16 \times 17 \times 18 = 1028160$

Question 2 [3 marks]

What is the last digit in the number represented by 7^{3122}? Show working.

2999

Pattern is: 7, 9, 3, 1 ✓
3122 ÷ 4 = 780.5 -> 2nd position ✓
∴ last digit is 9 ✓

$2999 \div 4 = 749.75$ - 3rd
∴ last digit is 3.

$7^1 = 7$
$7^2 = 49$
$7^3 = 343$
$7^4 = 2401$

7, 9, 3, 1

$7^{3000} \Rightarrow 1$
$7^{2999} \Rightarrow 3$

Question 3 [3 marks]

There is one triangle whose angle sizes are three square numbers. What are the sizes of its angles?

$7^5 = 16807$

16°, 64°, 100°
✓ ✓ ✓

Question 4 [3 marks]

There are three consecutive counting numbers a, b and c (in that order) such that:

$a^4 + b^4 = c^4 - (a-3)^3$

Find the numbers.

Using trial and error, the numbers are: 7, 8, 9 ✓✓

Question 5 [2, 3 = 5 marks]

A number is called a *picky number* if it is a positive integer and the number immediately before it and the number immediately after it are prime.

a) Explain clearly why a *picky number* must be an even number.

All primes are odd (except for 2) ∴ a number directly between two consecutive primes must be even, because odd and even numbers alternate. ✓

There is no possible number between the primes 2 and 3 ✓

b) Find all picky numbers between 1 and 30 inclusive.

Primes	Picky
2	
3	4
5	6
7	
11	12
13	
17	18
19	
23	
29	30
31	

4, 6, 12, 18, 30 ✓✓ −1 for each error

Question 6 [1, 2, 4 = 7 marks]
Here are the first five lines of a number pattern.

Line 1: $1^2 + 1^2 = 2$
Line 2: $1^2 + 2^2 = 5$
Line 3: $2^2 + 3^2 = 13$
Line 4: $3^2 + 5^2 = 34$
Line 5: $5^2 + 8^2 = 89$

1,
1,
2,
3,
5,
8,
13,
21,
34,
55,
89,
144
233
377
610
987

a) Give the next line of the number pattern

Line 6: $8^2 + 13^2 = 233$ ✓

b) Give the 10th line of the number pattern

Line 10: $55^2 + 89^2 = 10946$ ✓✓

c) Give a written description of the *n*th line of this number pattern.

The nth line will be: (nth Fibonacci number) $^2 + (F_{n+1})^2 = F_{2n+1}$

✓ ✓ ✓✓

Question 7 [3 marks]
Measure out exactly 4 cups of water from a pond using two pitchers (without markings); one holds exactly three cups and the other exactly five cups. Explain the process clearly.

- *Fill the 5C pitcher and pour into the 3C, leaving 2C*
- *Empty the 3C and pour the 2C into the 3C pitcher*
- *Fill the 5C pitcher and fill up the 3C pitcher (which has 1C capacity)*
- *You will have 4C left over in the 5C pitcher.*

✓✓✓

Question 8 [1, 6 = 7 marks]

JBA Health Clinic charges $500 for minor surgery and $750 for major surgery. Minor surgery lasts for 1 ½ hours and major surgery lasts for 2 ½ hours. The clinic has one operating room which is available for only 7 hours per day.

a) Which type of surgery earns the most money per hour?

Minor surgery - $333 ⅓ per hour ✓

b) The management wishes to use the operating room as much as possible whilst scheduling surgery to maximize revenue. Advise the management on the best way to make use of the operating room. Justify your advice by showing that it is the best option.

4 × minor and 0 × major = 6 hours and $2000 ✓
3 × minor and 2 × major = 7 hours and $2250 ✓✓
2 × minor and 2 × major = 8 hours -> too much

∴ 3 minor and 2 major would be best. ✓

Question 9 [3, 1, 1, 3, 1 = 9 marks]

The following table lists the numbers form 2 to 10, together with each number's factors, and the sum of these factors:

n	Factors of n	Sum of the factors
2	1, 2	3
3	1, 3	4
4	1, 2, 4	7
5	1, 5	6
6	1, 2, 3, 6	12
7	1, 7	8
8	1, 2, 4, 8	15
9	1, 3, 9	13
10	1, 2, 5, 10	18
11	*1, 11*	*12*
12	*1, 2, 3, 4, 6, 12*	*28*
13	*1, 13*	*14*
14	*1, 2, 7, 14*	*24*
15	*1, 3, 5, 15*	*24*
16	*1, 2, 4, 8, 16*	*31*
17	*1, 17*	*18*
18	*1, 2, 3, 6, 9, 18*	*39*
19	*1, 19*	*20*
20	*1, 2, 4, 5, 10, 20*	*42*

a) Complete the table for the numbers 11 to 20 ✓✓✓

b) A number n is called a *perfect number* if its factors add up to $2n$. From the table, write down all the perfect numbers, if any, between 2 and 20, inclusive.

6 ✓

c) After viewing the list of factors of the numbers from 2 to 10, Brandyn makes the following conjecture:
"The sum of the factors of any number n is less than or equal to $2n$."
Write down a number between 11 and 20, inclusive, which provides a counter-example to this conjecture:

12 or 18 or 20 ✓

d) Suppose that n is a perfect number, and let $q = 2n$. Would you expect the sum of the factors of q to be less than $2p$? Give a reason for your answer.

✓
No, they will sum to 2n plus extra ✓✓

e) What do the numbers in the table with an odd number of factors have in common?

Square numbers ✓

Question 10 [1, 3, 2, 3, 4 = 13 marks]

If n is a positive integer, the nth Mersenne number is defined to be the number: $M_n = 2^n - 1$

a) Complete the second column of the following table by calculating the values of M_3 and M_7.

n	M_n	Prime factors of M_n
1	1	-
2	3	3
3	*7* ✓	*7*
4	15	*3, 5* ✓✓
5	31	*31* ✓
6	63	*3, 7*
7	*127* ✓	*127*
8	255	3, 5, 7
9	511	7, 73
10	1023	3, 11, 31

-1 each error or omission

b) A prime number is a number that is divisible by only 1 and itself (Note that 1 is not prime). Write down all the prime factors of the Mersenne numbers M_3 and M_7 by completing the third column of the table.

c) For what values of n in the table is the Mersenne number M_n prime?

$n = 2, 3, 5, 7$ ✓✓

d) On seeing the completed table above, John makes the conjecture that if n is prime then the corresponding Mersenne number M_n is prime. We can check John's conjecture as follows:

i) Calculate M_{11}

$2^{11} - 1 = 2047$ ✓

ii) Is M_{11} prime? (Hint: Try to divide M_{11} by 23)

No, it is divisible by 23 ✓

iii) Is John's conjecture correct?

No ✓

e) In the light of the results of parts a) to d), which of the following statements is a reasonable conjecture about Mersenne numbers? If you think the statement is consistent with all the information given in parts a) to d), circle YES. If you think the statement is contradicted by the results of parts a) to d), circle NO.

i) All Mersenne numbers are prime YES / (*NO*) ✓

ii) If n is not prime then M_n is not prime (*YES*) / NO ✓

iii) If p is a prime factor of M_n, and k is any multiple of n, then p is also a prime factor of M_k

(*YES*) / NO ✓

iv) If p is a prime factor of M_n, and k any number greater than n, then p is also a prime factor of M_k YES / (*NO*) ✓

Question 11 [3 Marks]

If it takes six elves one hour to fill four sleighs with toys, how long will it take one elf to fill one sleigh?

6 elves take 60 minutes to fill 4 sleighs

1 elf takes 360 minutes to fill 4 sleighs

1 elf takes 90 minutes to fill 1 sleigh

Answer: 90 minutes ✓✓

Question 12 [2 marks]

I married my boyfriend, we had such fun; our happiness grew when we had a son. If our son's father is Roy's first boy, what relation am I, then to Roy?

His daughter-in-law ✓✓